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Archimedes

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Archimedes

Sheet 1

Solutions and comments

This programme of the Mentoring Scheme is named after Archimedes of Syracuse (287–212 BCE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Archimedes.html for more information.

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Version 1.1, Oct 2019

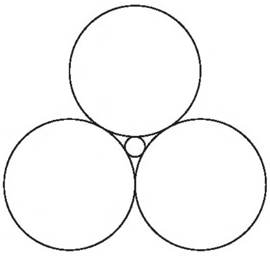
Archimedes was born and died in Syracuse on the island of Sicily. He said in one of his works that his father Phidias was an astronomer. It seems likely that he studied in Alexandria, Egypt with the successors of Euclid. He became known widely for his inventions of the engines of war, particularly for the ingenious devices that were used to fend off the Romans during the two year siege of Syracuse which ended in 212 BCE. The Roman author Plutarch and historian Livy provide information about Archimedes in their writings.

His death at the hands of a Roman soldier has generated a number of stories. One wonders what happened to the soldier(s) when the general Marcellus heard that Archimedes had been slain.

He was responsible for the method of exhaustion by which a shape is divided into increasingly thin slices by which the area or volume can be calculated, a process which was not advanced further until the days of Fermat followed by Newton and Leibnitz. He wrote works on cylinders, spheres, cones, centres of mass, levers, pulleys, floating bodies and how to deal with very large numbers. He found that $\frac{10}{71} < \pi < \frac{1}{7}$; in modern terms this is accurate to the 3rd decimal place. Other works are lost though referred to by later authors.

Interestingly, some of his work has survived. This is because parchment was valuable, and it could be scraped clean and reused. The original writing can sometimes be detected using modern scientific techniques. One of Archimedes' lost works was found in this way in a 13th century prayer book; this is now in the Walters Art Museum in Baltimore, USA. To read more about this, see http://www.archimedespalimpsest.org/

1. Three circles, each of radius 1 unit, touch each other externally as shown in the diagram. A small circle is placed in the middle so that it touches each of these circles. What is its radius?



ANSWER $\frac{2}{3}\sqrt{3} - 1$

SOLUTION

Let the centre of the small circle be O and the centres of the large circles be A, B and C.

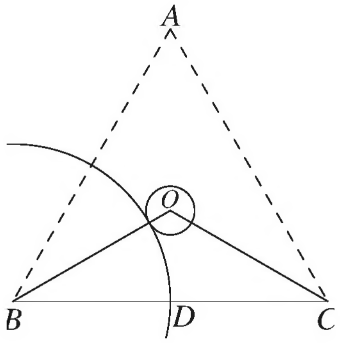
$\triangle ABC$ is equilateral. Its medians meet at O (by symmetry) and are perpendicular to the sides of $\triangle ABC$ (again by symmetry). We have $AB = 2$.

If D is the midpoint of BC, then $BD = 1$. Using Pythagoras's Theorem on $\triangle ABD$, $1^2 + AD^2 = 2^2$, hence $AD = \sqrt{3}$.

The medians split each other in the ratio 2 : 1 (see note G in the mentoring notes), so $OA = \frac{2}{3}\sqrt{3} = OB = OC$.

Alternatively, we can find the length OB by noting that $\triangle BOD$ has angles 30°, 60° and 90°, so $OB : BD = 2 : \sqrt{3}$; using $BD = 1$ gives us OB.

All we then have to do is subtract the radius of one of the large circles to find the radius of the small circle.



2. You are given that $2 + y = x^2$. Put $x = 2 + a$ and $y = 2 + b$, where both a and b are between 0 and 1. Show that if y is close to 2, then x is closer.

What about the case where x is a little less than 2?

Solution

We are considering the case where $0 < a < 1$ and $0 < b < 1$, so $x > 2$ and $y > 2$.

Substituting in $2 + y = x^2$ gives: $2 + 2 + b = (2 + a)^2$

Expanding gives: $4 + b = 4 + 4a + a^2$

so $b = a(4 + a)$

so $a = \dfrac{b}{4 + a}$

Since $4 + a$ is greater than 4, we see that $a < \frac{1}{4}b$, and hence x is closer to 2 than y is.

Now consider the case where $x < 2$ and hence $y < 2$ also. Write $x = 2 - a$ and $y = 2 - b$, where again both a and b are between 0 and 1. The algebra is similar:

Substituting in $2 + y = x^2$ gives: $2 + 2 - b = (2 - a)^2$

Expanding gives: $4 - b = 4 - 4a + a^2$

so $b = a(4 - a)$

so $a = \dfrac{b}{4 - a}$

This time, $4 - a$ is at least 3 (as $0 < a < 1$), so we have $a < \frac{1}{3}b$; thus in this case, x is again closer to 2 than y is.

3. Is it possible to draw a triangle in which the sum of every pair of angles is less than 120°? You must justify your answer.

Answer It is not possible

Solution

Let the angles be A, B and C. Suppose that $A + B < 120°$ and $B + C < 120°$ and $C + A < 120°$. Adding these three inequalities gives $2A + 2B + 2C < 360°$, so $A + B + C < 180°$, which clearly contradicts the angle sum rule for a triangle ($A + B + C = 180°$).

Alternatively, suppose that $B + C < 120°$. Then since the angles of a triangle sum to 180°, it follows that $A > 60°$. Similarly, as $A + C < 120°$, we have $B > 60°$. Adding these gives $A + B > 120°$, contrary to our assumption. (As a third alternative, we could show that $C > 60°$ in a similar way, and then added the three inequalities to obtain $A + B + C > 180°$.)

4. We choose six (different) numbers from the ten integers 1, 2, 3, . . . , 10. Prove that their product is divisible by a perfect square greater than 1.

SOLUTION

We aim to choose six different numbers with a product that is *not* divisible by a perfect square greater than 1.

If we choose two even numbers, then 4 is a factor of the product.

If we choose 9 as one of the numbers, then 9 is a factor of the product.

So we can choose at most one even number, and odd numbers from 1, 3, 5, 7. This gives us a maximum of five numbers, so it is impossible to choose six different numbers in this way. (Choosing 1, 2, 3, 5, 7 gives a product that is not divisible by a perfect square greater than 1.)

A more general approach arises from thinking about primes. There are 4 primes between 1 and 10. Every integer between 2 and 10 is divisible by at least one of these primes. Therefore if we select 5 or more integers in this range, at least two of them must be divisible by the same prime, and so their product will be divisible by the square of this prime. Thus, when we select 6 integers between 1 and 10, at least 5 of them must between 2 and 10. Hence their product must be divisible by a square.

> Can you generalise the question?

5. At the reading of a will, Prudence and Ravi are told that they will inherit one sixth of the total money between them. There are conditions, though. They are not allowed to inherit the same amount as each other, but each amount must be a simple fraction of the money, that is a fraction with numerator 1.

If Prudence inherits $\frac{1}{x}$ and Ravi inherits $\frac{1}{y}$ of the total money (where x and y are whole numbers), show that $xy - 6x - 6y = 0$.

In how many ways can the money be split between them?

ANSWER 8 ways

SOLUTION

Prudence inherits $\frac{1}{x}$ and Ravi inherits $\frac{1}{y}$; between them, they inherit $\frac{1}{6}$, so:

$$\begin{aligned} \frac{1}{x}+\frac{1}{y} &= \frac{1}{6} \\ 6x+6y &= xy && \text{multiplying both sides by } 6xy \\ xy-6x-6y &= 0 && \text{rearranging} \end{aligned}$$

Using the hint, we expand $(x-a)(y-a)$ to get:

$$(x-a)(y-a) \equiv xy - ax - ay + a^2.$$

(We use a three-line equals sign here to indicate that this is an *identity*: it is true whatever the values of x and y are.) Comparing this to the equation we have, if we choose $a = 6$, we have

$$(x-6)(y-6) \equiv xy - 6x - 6y + 36.$$

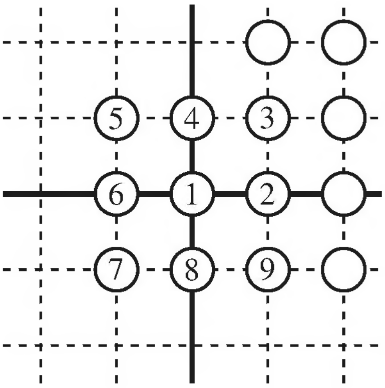
Now in our case, $xy - 6x - 6y = 0$, so $xy - 6x - 6y + 36 = 36$, hence $(x-6)(y-6) = 36$.

Since x and y must both be positive integers, we require $(x-6)$ and $(y-6)$ to be a pair of factors of 36, each greater than $1 - 6 = -5$. This gives the following possibilities:

$x-6$	$y-6$	x	y
1	36	7	42
2	18	8	24
3	12	9	18
4	9	10	13
6	6	(12)	(12)
9	4	13	10
12	3	18	9
18	2	24	8
36	1	42	7

Note that $x = 12$ and $y = 12$ is not an acceptable solution, as they are not allowed to inherit the same as each other. Also, the last four solutions are different from the first four only in that the amounts Prudence and Ravi receive are swapped.

6. The positive integers starting at 1 are written down on a graph at the points $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, $(-1,1)$, $(-1,0)$ and so on, forming an anti-clockwise spiral of integer grid points as shown in the diagram. Find the coordinates of the point where 2019 is written.



Answer $(16,-22)$

Solution

Every time the spiral completes a square at a 'bottom right corner', the number at that point will be a square number. At $(1,-1)$, it is $3^2 = 9$; at $(2,-2)$, the square has side length 5, so the number is $5^2 = 25$. We can generalise this to see that the first $(2n+1)^2$ points form a square with side $(2n+1)$ units, and the bottom right corner is at $(n,-n)$.

Now $2019 = 2025 - 6 = 45^2 - 6$. Because $45 = 2 \times 22 + 1$, 2025 will appear at $(22,-22)$. The number 2019 will appear 6 steps before this.

7. A triangle ABC has two vertices $A(-3,1)$ and $B(3,-1)$. The third vertex, C, is at $(2,k)$. Find all the values of k for which the triangle is isosceles.

Answer $k = 6, 1 \pm \sqrt{15}, -1 \pm \sqrt{39}$

Solution

There are three cases to consider, and these are shown in the three diagrams below; they depend on which side is the base of the isosceles triangle.

(a) AB is the base and $CA = CB$. Therefore C lies on the perpendicular bisector of AB, so passes through $(0,0)$. Because a point moving along AB drops 1 unit for every 3 moved to the right, a point on the perpendicular bisector rises 3 units for every 1 unit moved to the right. Hence C is at $(2,6)$ in this case. (You can also express this algebraically: the line AB has equation $y = -\frac{1}{3}x$, and the perpendicular bisector has equation $y = 3x$.)

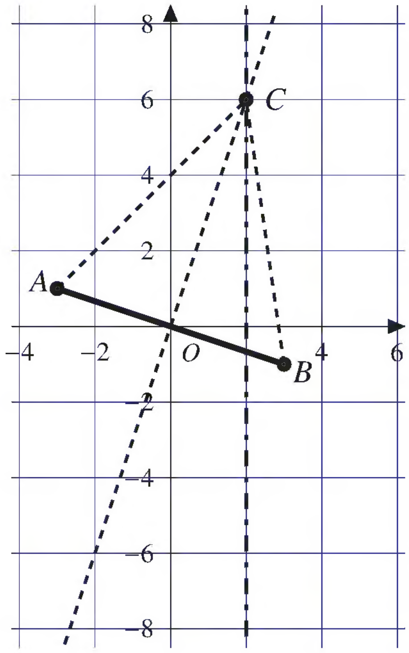
(b) BC is the base and $AB = AC$. We note that $AB = \sqrt{40}$, from Pythagoras's Theorem, so we require $AC = \sqrt{40}$ as well. Therefore C must line on a circle with radius $\sqrt{40}$ with its centre at A.

The line $x = 2$ meets this circle at two points, C_1 and C_2, as shown in the diagram. Let Q be the points $(2,1)$, so that AQ is perpendicular to the line $x = 2$. Then $AQ^2 + QC^2 = AC^2$.

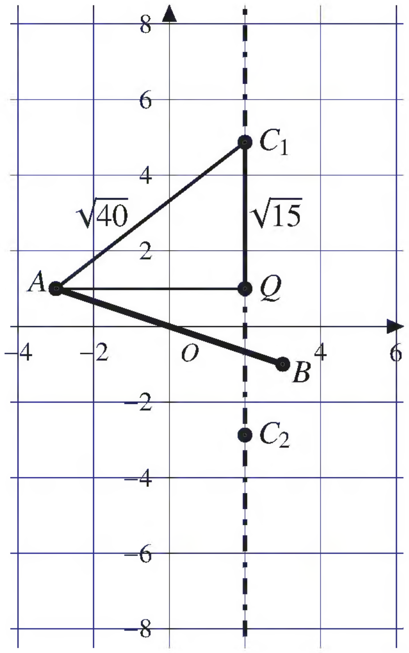
Now $AQ = 5$ and $AC = \sqrt{40}$, so by Pythagoras's Theorem, $CQ = \sqrt{15}$.

Hence C_1 is at $(2, 1 + \sqrt{15})$ and C_2 is at $(2, 1 - \sqrt{15})$.

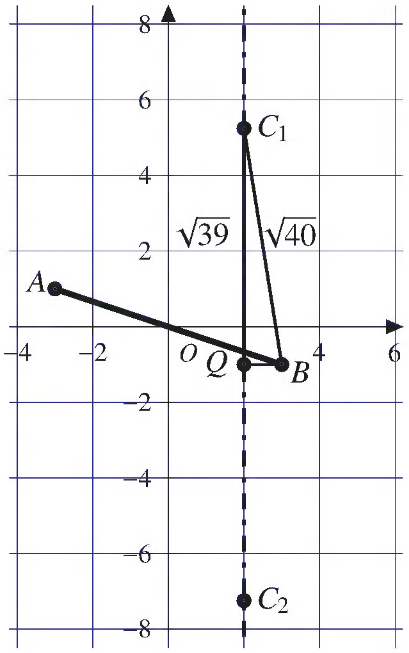
(c) AC is the base and $BA = BC$. This is very similar to the previous case, where we now require $BC = \sqrt{40}$. We take Q to be $(2,-1)$ so that BQ is perpendicular to $x = 2$, giving $CQ = \sqrt{39}$. Hence C_1 is at $(2, -1 + \sqrt{39})$ and C_2 is at $(2, -1 - \sqrt{39})$.



AB is base: $CA = CB$
line $y = 3x$ meets $x = 2$ at $(2,6)$



BC is base: $AB = AC$
$AQ = 5$



AC is base: $BA = BC$
$BQ = 1$

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Archimedes

Sheet 2

Solutions and comments

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1. Magminimus wasn't very good at arithmetic. He always got confused between numbers like XC and CX. His father Magmaximus, a senator in Ancient Rome, was an expert. He had become a very rich man trading throughout the Mediterranean, and could afford to keep lots of slaves including a very strict tutor for his son. The tutor arranged 11 numbers around a circle and asked Magminimus to work out the differences between adjacent numbers. (People did not use negative numbers at the time so simply subtracted the smaller from the larger.) The boy gave 4 Is, 4 IIs and 3 IIIs as his answer. His tutor was quick to reproach him disapprovingly. Prove that Magminimus made a mistake.

SOLUTION

Let the numbers be $a_1, a_2, \dots, a_{11}$. Then the signed differences (allowing negative answers) are $a_1 - a_2, a_2 - a_3, \dots, a_{11} - a_1$. Negating any of these differences as Magminimus would have done in order to get a non-signed (positive) difference does not change whether a difference is even or odd. Since the sum of the signed differences is 0, which is even, the sum of the non-signed differences must also be even. This is not what Magminimus gave as an answer.

Note that we do not know what is wrong with Magminimus's solution. All we know is that he should have found an even number of odd differences.

2. Simplify $(a+b)^2+(a-b)^2$. Hence show that in a right angled triangle with shorter sides of length a and b and hypotenuse of length c, it will always be true that $a + b \leq \sqrt{2}\,c$.

SOLUTION

We work out that $(a + b)^2 + (a - b)^2 = (a^2 + 2ab + b^2) + (a^2 - 2ab + b^2) = 2a^2 + 2b^2$.

Since c is the hypotenuse, applying Pythagoras's Theorem then shows that $(a+b)^2+(a-b)^2 = 2c^2$.

Every square number is 0 or positive so $(a - b)^2 \geq 0$. Hence $2c^2 \geq (a + b)^2$.

Because $a > 0$, $b > 0$ and $c > 0$ for a triangle, taking the positive square root completes the proof.

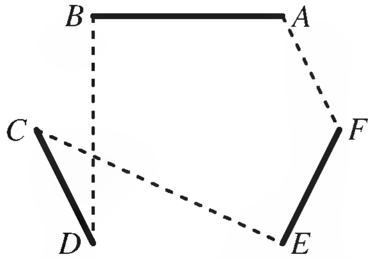
3. One Saturday, a tennis coach holds a training session for the six school pupils in his school's county school doubles league team. He is interested in seeing how each player adapts to playing alongside one of her team-mates. He pairs the players off at random to form three 'strings'. In the morning he puts the first string against the second to play one set, then puts the second string against the third and finally the third against the first.

In the afternoon he mixes up the pupils. No pupil plays with the same partner that she did in the morning. Given the choice he made in the morning, in how many ways can he pair up the pupils in the afternoon?

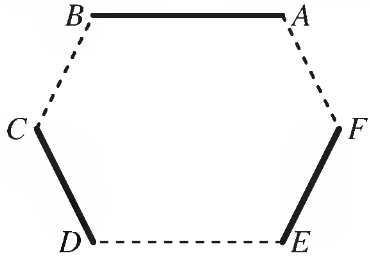
ANSWER 8

SOLUTION

The hexagon idea suggested in the hint can be made clearer by marking the sides which join morning pairings (say A with B, C with D and E with F) with thick lines. Such joins must be avoided in the afternoon. You then use diagonals or the unused sides to represent the afternoon pairings, for example:



Let us pick one player, say A; she can be joined to any of the four players C, D, E or F. B cannot be joined to the morning partner of this player, because then the remaining two players will be left as a pair. But B can be joined to either of the other two. There are thus 4 × 2 = 8 possible afternoon pairings. The above diagram showed the case where A is joined to F and B is joined to D; the following diagram shows the other case in which A is joined to F.



4. Show there there is a set of three consecutive odd integers, a, b and c, such that $a^2+b^2+c^2$ is a four digit integer with all the digits equal.

ANSWER 41, 43, 45

SOLUTION

Here is a method using some algebra. Let $a = 2n - 1$, $b = 2n + 1$ and $c = 2n + 3$. (This makes the algebra simpler than taking $a = 2n + 1$, $b = 2n + 3$ and $c = 2n + 5$, as the numbers are not as large.) We multiply out and simplify:

$$\begin{aligned} a^2 + b^2 + c^2 &= (2n-1)(2n-1) + (2n+1)(2n+1) + (2n+3)(2n+3) \\ &= 4n^2 - 2n - 2n + 1 + 4n^2 + 2n + 2n + 1 + 4n^2 + 6n + 6n + 9 \\ &= 12n^2 + 12n + 11 \end{aligned}$$

There are various features to be noticed. In order to have a four digit integer, $9 \le n \le 28$. 11 less than the required number, namely $12n^2 + 12n$, must be of the form $xxyy_{10}$ and must be even. (Why can there not be a carry from the hundreds column?) It must also be divisible by 3 so the only candidate for the value of $12n^2 + 12n$ is 5544. Since the squares are all approximately equal, we can find the middle square by dividing by 3: $5544 \div 3 = 1848$, and $\sqrt{1848} \approx 43$.

We can then check: $41^2 + 43^2 + 45^2 = 1681 + 1849 + 2025 = 5555$ and we are done.

Here is an alternative which simplifies the algebra by just assuming n is odd, and taking $a = n - 2$, $b = n$ and $c = n + 2$. This gives

$$\begin{aligned} a^2 + b^2 + c^2 &= (n-2)(n-2) + n^2 + (n+2)(n+2) \\ &= n^2 - 2n - 2n + 4 + n^2 + n^2 + 2n + 2n + 4 \\ &= 3n^2 + 8 \end{aligned}$$

Since n is odd, this whole expression is odd. The only possible values for such a four digit sum are 1111, 3333, 5555, 7777 or 9999. On subtracting 8 from each of these, only $5555 - 8 = 5547$ is a multiple of 3 as required by the expression. We now see that $5547 \div 3 = 1849 = 43^2$.

5. Katie thinks she has come up with a hard problem with which to challenge her classmates. She pretends that she has a pack of cards numbered in order from 1 to 10 000. She tells the class that on the first step they must imagine that she removes all the cards showing a square number. Then she will renumber the N cards still left in the pack from 1 to N. She says that this step can be repeated over and over with N getting smaller after each step. Her challenge is for the class to find out how many steps are needed until there is just one card remaining. What answer does she expect?

ANSWER 198

SOLUTION

100 cards are removed on the first step leaving 9900 cards. The largest square not exceeding 9900 is $9801 = 99^2$. Hence 99 cards can be removed on the second step leaving exactly 9801 cards. Since this is a perfect square we can continue the process 2 steps at a time where on every second step N is the next smaller perfect square. We need 99 pairs of steps to get from $N = 100^2$ to $N = 1^2$.

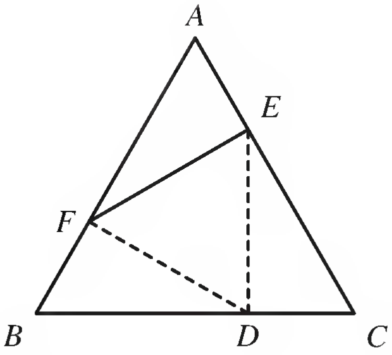
An algebraic argument runs as follows. Suppose we start with n^2 cards. After the first step we have $n^2 - n$ cards. This is greater than $(n-1)^2 = n^2 - 2n + 1$, so we remove $n - 1$ cards on the next step. After two steps we have $n^2 - n - (n-1)$ which is exactly $(n-1)^2$ cards.

> What would happen if the starting number is not a square?

6. $\triangle ABC$ is equilateral and has sides of length 6. D, E and F are on BC, CA and AB respectively so that $CD = AE = 2$ and $\angle AEF = 90°$. What is the (exact) area of $\triangle DEF$?

ANSWER $3\sqrt{3}$

SOLUTION



$CE = 4$ so that $CE = 2 \times CD$. $\angle DCE = 60°$.

Thus $\triangle CDE$ is half an equilateral triangle with $CDE = 90°$.

$\triangle AEF$ is congruent to $\triangle CDE$ because $CD = AE$, $\angle AEF = \angle CDE = 90°$ and $\angle DCE = \angle EAF$.

Hence $AF = 4$ and $BF = 2$.

Thus $\triangle BFD$ is congruent to these other two triangles by SAS (see section D of the notes for mentees). It follows from this that $DE = EF = FD$ and so the triangle $\triangle DEF$ is equilateral.

We can use Pythagoras's Theorem to find DE: we have $EC^2 = CD^2 + DE^2$, giving

$$4^2 = 2^2 + DE^2$$

$$\text{so} \quad DE = \sqrt{4^2 - 2^2} = \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}. \qquad (1)$$

There are several ways to proceed from this point.

A first is to calculate the area of $\triangle DEF$ directly, using the fact that it is equilateral. It's base is $2\sqrt{3}$, so if its height is h, we must have:

$$h^2 + (\tfrac{1}{2} \times 2\sqrt{3})^2 = (2\sqrt{3})^2$$

$$\text{so} \quad h^2 + 3 = 12$$

$$\text{so} \quad h = 3$$

Therefore the area of $\triangle DEF$ is $\frac{1}{2} \times 2\sqrt{3} \times 3 = 3\sqrt{3}$.

Alternatively we can proceed from equation (1) to work out the area of $\triangle CDE$: it is $\frac{1}{2} \times 2 \times DE = 2\sqrt{3}$. Also, the area of $\triangle DEF$ is the area of $\triangle ABC$ minus three times the area of $\triangle DEF$.

There are various ways of proceeding from this point. One method is to calculate the area of $\triangle ABC$ in the same way as we calculated the area of $\triangle DEF$ above. Another is to note that half of $\triangle ABC$ is similar to and $\frac{3}{2}$ times larger than $\triangle CDE$. Hence its area is $(\frac{3}{2})^2$ times the area of $\triangle CDE$, that is $(\frac{3}{2})^2 \times 2\sqrt{3} = \frac{9}{2}\sqrt{3}$, so the area of $\triangle ABC$ is twice this, $9\sqrt{3}$, and the area of $\triangle DEF$ is $9\sqrt{3} - 3 \times 2\sqrt{3} = 3\sqrt{3}$.

7. (a) An arc of a circle is drawn between diagonally opposite vertices of a rectangle comprising square tiles arranged in m rows of n tiles each. If the arc lies entirely within the rectangle and does not pass through any corners of the tiles *within* the rectangle, how many tiles does arc pass through? Will this still work if the tiles are rectangular and not square?

(b) A builder constructs a circular patio from a grid of 0.5 m square paving stones; the patio is centred on the corner of one of the paving stones. The radius of the patio is 3 m. How many paving stones does the builder have to cut to shape to form the patio?

(c) Will this result still apply if the diagonally opposite points of the rectangle are joined by a straight line rather than a circle? What conditions would have to apply to the values of m and n?

ANSWER (a) $m + n - 1$ tiles; (b) 44; (c) works provided HCF of m and n is 1

SOLUTION

(a) Less us suppose that the arc goes from the bottom left to the top right of the rectangle. Then as we go along the arc, we are always going rightwards and upwards. It therefore crosses exactly $m - 1$ row boundaries and $n - 1$ column boundaries (as it never crosses a boundary twice). Thus it enters $(m - 1) + (n - 1) = m + n - 2$ tiles, and so passes through $m + n - 1$ tiles, including the corner one it begins in. The tiles could be rectangular and the result would still be true.

(b) Consider just one quarter of the patio: the boundary crosses a 6 by 6 square of paving stones. From the result of part (a), this gives 11 paving stones to be split. (A little calculation using Pythagoras's Theorem will also convince you that this path does not pass through the corner of any paving stone. This would not be the case, though, if the radius were 2.5 m with the same paving stones.) There are four quarters of patio, giving 44 paving stones in total.

(c) There must be no integer greater than 1 which divides into both m and n. If, for example, m and n were both divisible by 3, then there would be (at least) two places inside the rectangle where the straight line passes through the corner of a tile. In general the line passes through $m + n - d$ spaces, where d is the highest common factor of m and n.

8. You are given a 50 by 50 board composed of 2500 squares which are initially all coloured white. At any *step*, a whole row or column of squares can change colour from white to black or black to white. Thus if 32 of the squares in a row are white and the other 18 are black, a *step* will change the white ones to black and the black ones to white, giving 32 black and 18 white squares. You aim to perform a sequence of *steps* so that you finish with exactly 2018 black squares. Is this possible?

ANSWER Yes, and there are two fundamentally different ways to do so

SOLUTION

Consider performing *steps* on m rows and n columns. For the m rows, all the squares in these rows will turn black except for the mn squares where the columns cross these rows. Hence there will be $50m - mn$ black squares by this process. We must also account for the squares which turn black in the n columns and which are not in the rows which have been turned. For these n columns, all the squares in these columns will turn black except for the mn squares where the rows cross these columns, giving a further $50n - mn$ black squares.

For the moment, replace our target 2018 with y so that we don't get overwhelmed with specific numbers, and can then generalise later. This gives us the equation

$$(50m - mn) + (50n - mn) = y.$$

Rearranging this we obtain

$$mn - 25m - 25n = -\tfrac{1}{2}y.$$

A standard technique for dealing with this type of equation, where the variables have to be integers, is to note that $(m - a)(n - b) = mn - an - bm + ab$. So adding $25 \times 25 = 625$ to both sides, we get

$$(m - 25)(n - 25) = 625 - \tfrac{1}{2}y.$$

In the case of $y = 2018$, the right hand side equals -384. This can be expressed as -3×2^7 and it has 16 factor pairs, namely

$$\begin{gathered}(-1, 384);\ (-2, 192);\ (-4, 96);\ (-8, 48);\ (-16, 24);\ (-32, 12);\ (-64, 6);\ (-128, 3);\\ (-3, 128);\ (-6, 64);\ (-12, 32);\ (-24, 16);\ (-48, 8);\ (-96, 4);\ (-192, 2);\ (-384, 1).\end{gathered}$$

Here, the pairs in the second row are almost the same as those in the first row, except that the negative sign appears on the other factor.

One of the pair is $m - 25$ and the other $n - 25$. Since m lies between 0 and 50, $m - 25$ must lie between -25 and 25, and similarly for $n - 25$. Therefore the factor pairs that give allowable values are $(-16, 24)$ and $(-24, 16)$

Hence there are two types of solution, one with $m = 9$, $n = 49$ and the other with $m = 1$, $n = 41$ (or vice versa).

> You could consider which years between 2018 and 2100 could be substituted for y and still give a solution.

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Archimedes

Sheet 3

Solutions and comments

This programme of the Mentoring Scheme is named after Archimedes of Syracuse (287–212 BCE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Archimedes.html for more information.

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Version 1.1, Dec 2019

1. A football team competes in a league in which each victory is worth 3 points, each draw 1 point and each loss 0 points. After the first 13 matches the team has 29 points and has lost as many games as it has tied. How many matches has the team won so far?

ANSWER 9

SOLUTION

Let x be the number of tied matches (which equals the number of lost matches). Then the number of matches won is $13 - 2x$, and the number of points gained is

$$\begin{aligned} & 3(13-2x)+x+0=29 \\ \text{giving} \quad & 39-5x=29 \\ \text{so} \quad & x=2. \end{aligned}$$

Hence 27 points are gained from the matches won.

2. Prove that $2\sqrt{2} < \sqrt{10} < \sqrt{5} + \sqrt{3}$.

SOLUTION

It is easier to work with each inequality separately. Consider the left pair of numbers. If we square $2\sqrt{2}$ we obtain $4 \times 2 = 8$ which is clearly less than $(\sqrt{10})^2 = 10$; therefore, square rooting $8 < 10$, we get $2\sqrt{2} < \sqrt{10}$.

Next, consider the right pair. Again, the square of $\sqrt{10}$ is 10. The square of the right hand side is

$$\begin{aligned} (\sqrt{5}+\sqrt{3})^2 &= 5 + 2 \times \sqrt{5} \times \sqrt{3} + 3 \\ &= 8 + 2 \times \sqrt{15} \\ &> 10 \end{aligned}$$

because $\sqrt{15} > 1$. Thus $10 < (\sqrt{5}+\sqrt{3})^2$ and so $\sqrt{10} < \sqrt{5} + \sqrt{3}$ as we wished to show.

3. An integer N has first digit 4 and last digit 9. Between the first digit and the last digit are a number of 4s and then the same number of 8s (for example, 44448889). Prove that N is a perfect square.

SOLUTION

First you might have tested out the proposition and verified that $49 = 7^2$, $4489 = 67^2$, $444889 = 667^2$ and so on. How can we prove that this pattern continues indefinitely? We offer three different approaches.

Approach 1

Suppose that $x^2 = 44\ldots4488\ldots889$, where we do not assume x to be an integer (because we have not yet proved that the number is a square). Then $x^2 - 1 = 44\ldots4488\ldots888$.

But $x^2 - 1 = (x+1)(x-1)$. Our belief is that we should have $x = 66\ldots67$, so we consider the numbers $66\ldots68$ and $66\ldots66$ as $x+1$ and $x-1$. We can then calculate:

$$66\ldots68 \times 6 = 400\ldots08, \quad 66\ldots68 \times 60 = 4000\ldots80, \quad \ldots$$

Adding these together gives $66\ldots68 \times 66\ldots66 = 44\ldots4488\ldots88$, so that $x = 66\ldots67$ is the square root of $44\ldots4488\ldots889$, as we wanted to show.

For the next two approaches, we note that $\frac{1}{9}(10^n - 1) = 11\ldots11$, where there are n ones.

Approach 2

Suppose there are n 4s, $n-1$ 8s and a 9 in the integer N. We note that $66\ldots67$ (with n digits) is the nearest integer to $\frac{2}{3}$ of 10^n. More precisely, we have

$$66\ldots67 = \tfrac{2}{3} \times 10^n + \tfrac{1}{3}$$

Squaring this gives

$$\begin{aligned}
66\ldots67^2 &= (\tfrac{2}{3} \times 10^n + \tfrac{1}{3})^2\\
&= \tfrac{4}{9} \times 10^{2n} + 2 \times \tfrac{2}{3} \times 10^n \times \tfrac{1}{3} + \tfrac{1}{9}\\
&= \tfrac{4}{9} \times (10^{2n} - 1) + \tfrac{4}{9} + \tfrac{4}{9} \times (10^n - 1) + \tfrac{4}{9} + \tfrac{1}{9}\\
&= \underbrace{44\ldots44}_{2n \text{ digits}} + \underbrace{44\ldots44}_{n \text{ digits}} + 1\\
&= 44\ldots4488\ldots89 = N
\end{aligned}$$

Alternatively, we could have calculated like this:

$$\begin{aligned}
66\ldots67^2 &= (\tfrac{2}{3} \times 10^n + \tfrac{1}{3})^2\\
&= \tfrac{4}{9} \times 10^{2n} + 2 \times \tfrac{2}{3} \times 10^n \times \tfrac{1}{3} + \tfrac{1}{9}\\
&= \tfrac{4}{9} \times 10^{2n} + \tfrac{4}{9} \times 10^n + \tfrac{1}{9}\\
&= \tfrac{4}{9}(10^{2n} - 10^n) + \tfrac{8}{9}(10^n - 1) + 1\\
&= \tfrac{4}{9}(10^n - 1) \times 10^n + \tfrac{8}{9}(10^n - 1) + 1\\
&= 44\ldots44 \times 10^n + 88\ldots88 + 1 = N
\end{aligned}$$

Approach 3

We can avoid fractions by writing

$$\underbrace{44\ldots44}_{n\text{ 4s}}\underbrace{88\ldots88}_{n\text{ 8s}} = \underbrace{11\ldots11}_{n\text{ 1s}} \times 4\underbrace{0\ldots0}_{n-1\text{ 0s}}8.$$

Let $a = 11\ldots11$, where there are n ones. Then $9a + 1 = 10^n$.

We can now express $44\ldots4488\ldots88$ algebraically.

$$\begin{aligned}44\ldots4488\ldots88 &= 4a\times 10^n + 8a\\ &= 4a(9a+1)+8a\\ &= 36a^2+12a\\ \text{so}\quad 44\ldots4488\ldots89 &= 36a^2+12a+1\\ &= (6a+1)^2\end{aligned}$$

which proves that $44\ldots4488\ldots89 = 66\ldots67^2$ is a square.

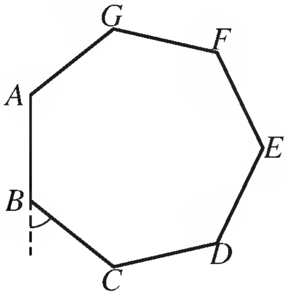
4. A heptagon has 7 sides. What is the interior angle of a regular heptagon?

Two regular heptagons $ABCDEFG$ and $ABPQRST$ are placed back to back so that AB is an axis of symmetry for the figure and also a common side. Show that EBQ is a straight line.

ANSWER $128\frac{4}{7}^\circ$

SOLUTION

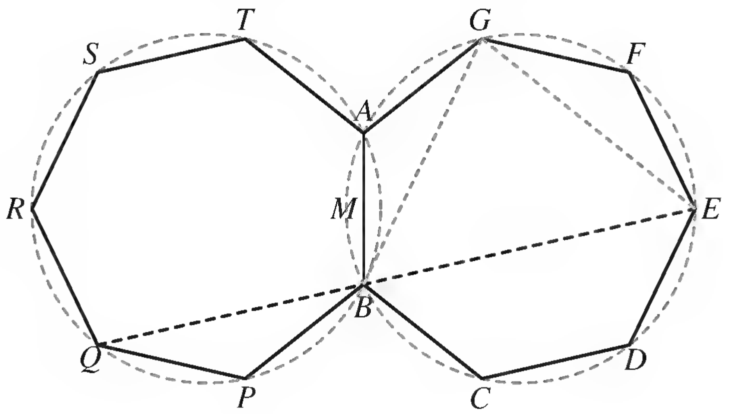
In the following diagram, the *exterior angle* at B is the angle between CB and the side AB extended, as marked. The other exterior angles of this heptagon are similar: they are the angle between one side and an adjacent side extended.



Let us call this exterior angle x. Since all seven exterior angles of a regular heptagon are equal, $x = \frac{360^\circ}{7}$. Therefore

$$\text{interior angle} = \angle ABC = 180^\circ - \frac{360^\circ}{7} = 128\tfrac{4}{7}^\circ.$$

Now moving onto the second part of the question, we have drawn the two back-to-back heptagons. We have also drawn circles around the heptagons to emphasise the regularity of the figure; they might also be helpful to prove the result.



We must be very careful not to assume that EBQ is a straight line anywhere within the argument. It is surprisingly easy to make an error like this! In this case, we could show that QB is parallel to BE, or that $\angle QBA + \angle QBE = 180°$, for example.

Approach 1

As one approach, we can see by symmetry that AF is parallel to BE, and also parallel to QB by rotating either heptagon through 180° onto the other about M, the midpoint of AB.

We can make this more precise: the pentagon $ABCDEFG$ has a line of symmetry passing through G and the midpoint of CD. Since A and F reflect onto each other, as do B and E, the lines AF and BE are both perpendicular to the line of symmetry and hence parallel to each other. Likewise, the lines AR and BQ are parallel to each other.

Now the angles $\angle BAR$ and $\angle ABE$ are equal, by congruence of the two heptagons (or by rotating one of them about the point M). Therefore AR and BE are parallel by alternate angles, and so BQ and BE are parallel. Thus EBQ is a straight line.

Approach 2

As an alternative approach, we can work out the actual angles. The angles are awkward numbers, so it is clearer to use variables. Consider first $\triangle BPQ$. Let $\angle PBQ = \alpha$. Because all the sides of a regular heptagon are equal, this triangle is isosceles: $BP = PQ$ and $\angle PQB = \alpha$. Hence the $\angle BPQ = 180° - 2\alpha$. (If we compare this to our earlier result, that $\angle BPQ = 180° - \frac{360°}{7}$, we find $\alpha = \frac{180°}{7}$, though we do not need this information.)

We can now calculate

$$\angle ABQ = \angle ABP - \angle PBQ = (180° - 2\alpha) - \alpha = 180° - 3\alpha.$$

We now need to calculate $\angle ABE$; if we can show that $\angle ABE = 3\alpha$, this will prove that EBQ is a straight line. We hope it will come to 3α in order to prove that QBE is a straight line. Here is one way we might tackle it.

Because a regular heptagon is symmetric, the triangles $\triangle GAB$ and $\triangle GFE$ are congruent. (They are also both congruent to $\triangle BPQ$ which we analysed above.) Hence $GB = GE$ so $\triangle GBE$ is isosceles.

We can also calculate $\angle BGE = \angle AGF - \angle AGB - \angle FGE = (180° - 2\alpha) - \alpha - \alpha = 180° - 4\alpha$.

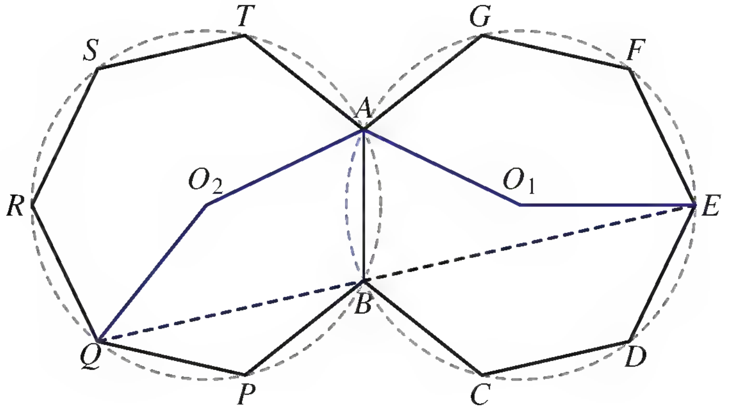
Therefore $\angle GBE = \frac{1}{2}(180^\circ - (180^\circ - 4\alpha)) = 2\alpha$.

Thus $\angle ABE = \angle ABG + \angle GBE = \alpha + 2\alpha = 3\alpha$, and $\angle ABQ + \angle ABE = 180^\circ$, so EBQ is a straight line.

Approach 3

Using the circles sketched in the diagram, we can use one of the circle theorems to answer this question (see section H of the https://www.ukmt.org.uk/enrichment/mentoring-scheme on the UKMT website).

The specific theorem of relevance here is that the angle at the centre of a circle is twice the angle at the circumference.



Using this theorem with this figure, we have $\angle AO_2Q = 2\angle ABQ$, where the first angle is a reflex angle. Now $\angle AO_2Q$ is $\frac{4}{7}$ of a complete circle, so $\angle ABQ = \frac{4}{7} \times 180^\circ$. Similarly, $\angle AO_1E = 2\angle ABE$, and as $\angle AO_1E$ is $\frac{3}{7}$ of a complete circle, $\angle ABE = \frac{3}{7} \times 180^\circ$. Thus $\angle ABQ + \angle ABE = 180^\circ$ and EBQ is a straight line.

> Because there are so many links between angles in the figure, no one proof is necessarily "better" than another. However, some proofs may be more compact than others.
>
> We reiterate that it is crucial that you do not assume the result you are trying to prove as part of the argument.

5. Can the numbers

$$\frac{14n+5}{9} \quad \text{and} \quad \frac{17n-5}{12}$$

both be integers for some integer n?

ANSWER No

SOLUTION

Approach 1

We can simplify the two fractions:

$$\begin{aligned}\frac{14n+5}{9} &= \frac{9n+5n+5}{9}\\ &= n+\frac{5(n+1)}{9}\end{aligned}$$

and

$$\begin{aligned}\frac{17n-5}{12} &= \frac{12n+5n-5}{12}\\ &= n+\frac{5(n-1)}{12}\end{aligned}$$

If both of these are integers, then $n+1$ must be divisible by 9 and $n-1$ must be divisible by 12. This requires both $n+1$ and $n-1$ to be divisible by 3. This is impossible since the difference between these two numbers is 2, which is not a multiple of 3.

Approach 2

If we let $\frac{14n+5}{9} = a$ and $\frac{17n-5}{12} = b$, where both a and b are integers, then we can rewrite these as

$$\begin{aligned}14n+5 &= 9a\\ 17n-5 &= 12b.\end{aligned}$$

Adding these gives $31n = 9a + 12b$. The right hand side is a multiple of 3, so the left hand side must be too. Since 3 is prime and 31 is not a multiple of 3, n must be a multiple of 3.

But then consider the equation $14n + 5 = 9a$: the left hand side is not a multiple of 3 (as $14n$ is a multiple of 3), but the right hand side is. This is impossible, so we cannot have such integers a and b.

Approach 3

Using modular arithmetic, the first equation simplifies as follows:

$$\begin{aligned}&& 14n+5 &\equiv 0 \pmod 9\\ \text{so} && 5n+5 &\equiv 0 \pmod 9\\ \text{so} && n &\equiv 8 \pmod 9,\end{aligned}$$

hence $n = 9k + 8$ for some integer k.

We also require $17n - 5 \equiv 0 \pmod{12}$ from the second fraction, so substituting this formula for n, we require

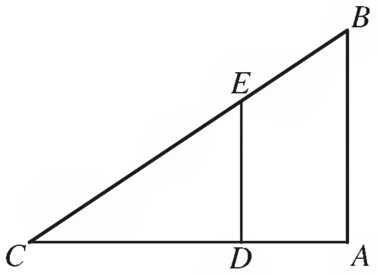
$$\begin{aligned} & 17(9k+8) - 5 \equiv 0 && \pmod{12} \\ \text{so} \quad & 5(9k+8) - 5 \equiv 0 && \pmod{12} \\ \text{giving} \quad & 45k + 35 \equiv 0 && \pmod{12} \\ \text{so} \quad & 9k + 11 \equiv 0 && \pmod{12} \\ \text{thus} \quad & 9k \equiv 1 && \pmod{12} \end{aligned}$$

But this equation has no solution, and so there is no value of n for which both the original fractions are integers.

6. In the diagram, DE is parallel to AB, $CD = AB$ and $\angle BAC = 90°$.

Furthermore, the area of $\triangle DEC$ is $\frac{1}{3}$ of that of $\triangle ABC$.

Show that $\angle BCA = 30°$.



Solution

Since DE is parallel to AB, we have $\angle CDE = \angle CAB = 90°$ and $\angle CED = \angle CBA$.

Hence $\triangle DEC$ is similar to $\triangle ABC$. Let $AB = CD = 1$ and $AC = x$; this amounts to choosing a measurement scale and does not affect any angles. We start by finding AC, and we show two ways to do so.

Approach 1

Then the area of $\triangle ABC = \frac{1}{2}x$.

As $\triangle DEC$ is similar to $\triangle ABC$, we have $DE : CD = AB : AC$. Hence

$$\frac{DE}{1} = \frac{1}{x}$$
$$\text{so area of } \triangle DEC = \frac{1}{2} \times \frac{1}{x}$$

But this area is $\frac{1}{3}$ of the area of $\triangle ABC$, which gives

$$\frac{1}{3} \times \frac{1}{2}x = \frac{1}{2} \times \frac{1}{x}$$

which we can rearrange to $x^2 = 3$.

Approach 2

We are given the ratio area of $\triangle ABC$: area of $\triangle DEC = 3 : 1$. Since the ratio of areas of similar figures is the ratio of the *squares* of corresponding lengths, this ratio is also $AC^2 : CD^2$. Hence $AC^2 = 3$.

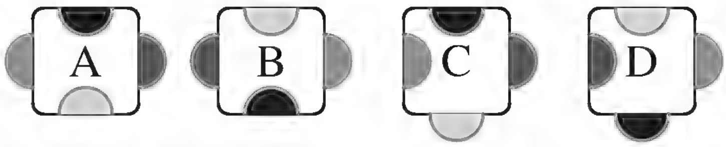
We now proceed to use the value of AC to prove what is required. We have $AB = 1$ and $AC = \sqrt{3}$. You may know enough trigonometry to give the solution directly from here. Alternatively, we can argue as follows.

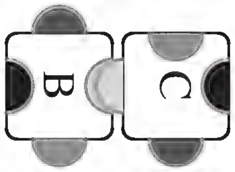
By Pythagoras's Theorem on $\triangle ABC$, $BC^2 = AC^2 + 1^2 = 4$. Thus $BC = 2 = 2 \times AB$.

We now extend BA to F so that $BF = 2$. A is the midpoint of BF, and we are given that $\angle BAC = 90°$. This proves that $\triangle BCF$ is equilateral. Hence $\angle BCA = 30°$.

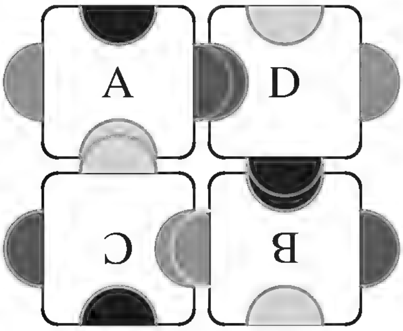
7. A child's puzzle consists of four square cards as shown below, each of which has two tabs and two slots. When two cards are placed next to each other, the tab must have the same shade as the slot over which it fits. An example is shown on the right. Ignoring rotated versions, show that there are only two ways to arrange the four cards in a 2 by 2 square.

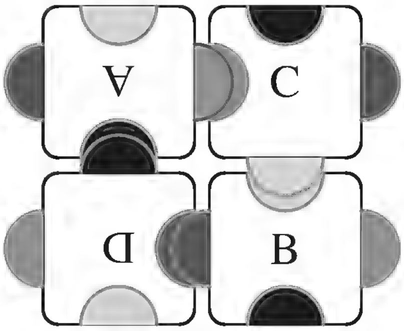
Unused tabs may project outside the finished square.





Solution





There are only these two possibilities for the following reasons.

First, cards A and B are the same and cannot be next to each other, therefore they must be diagonally opposite. Hence C and D are also diagonally opposite each other.

Now A and B have two mid grey and two dark grey tabs, and two white and two black slots. Thus the black tab and dark grey slot on D, and the mid grey slot and white tab on C must be used. There are only two ways to achieve this (up to rotation), as shown in the diagrams above.

8. Find all the triples of positive integers (a, b, n) that satisfy the equation $a! + b! = 2^n$.

Note that $a!$ is called 'a factorial', and it means all of the whole numbers up to a multiplied together. For example, $4! = 4 \times 3 \times 2 \times 1$. This is also known as 'factorial 4', and sometimes read as '4 bang' or '4 shriek'.

ANSWER $(1, 1, 1)$, $(2, 2, 2)$, $(2, 3, 3)$ and $(3, 2, 3)$

SOLUTION

We will assume that $b \geq a$. We can always switch round the values of a and b to give this order without altering the value of the left hand side. We consider the different possible values of a.

If $a \geq 3$, then $a!$ is divisible by 3 and b is also divisible by 3. Their sum is therefore divisible by 3 and so cannot be equal to a power of 2. So we must have $a = 1$ or $a = 2$.

If $a = 2$, we now consider possible values of $b \geq a$. If $b = 2$, then the left hand side is $2! + 2! = 4 = 2^2$, so we get the solution $(2, 2, 2)$. If $b = 3$, then the left hand side is $2! + 3! = 8 = 2^3$, so we get the two solutions $(2, 3, 3)$ and $(3, 2, 3)$ (swapping a and b). If $b \geq 4$, then there are no solutions: $b!$ is divisible by 4, but then $a! + b! = 2 + 4k = 2(1 + 2k)$ is not a power of 2.

Finally, consider $a = 1$. Since $a! = 1$ is odd, we require $b!$ to also be odd, since 2^n is even when n is a positive integer. Only $b = 1$ works, giving the final solution $(1, 1, 1)$.

> In order to maintain the property that $n(n - 1)! = n!$ when $n = 1$, we take $0! = 1$ and do not define $n!$ for negative integers. If we had allowed a, b, and c to be non-negative integers, we would obtain the further solutions $(0, 0, 1)$, $(0, 1, 1)$ and $(1, 0, 1)$, all of which give $1 + 1 = 2$.

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Archimedes

Sheet 4

Solutions and comments

This programme of the Mentoring Scheme is named after Archimedes of Syracuse (287–212 BCE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Archimedes.html for more information.

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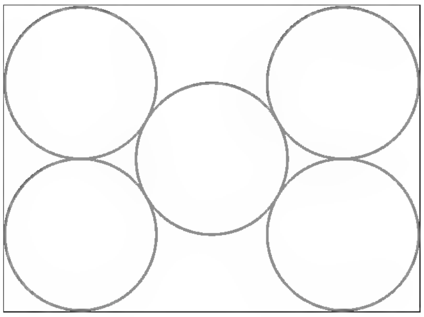
Version 1.1, Jan 2020

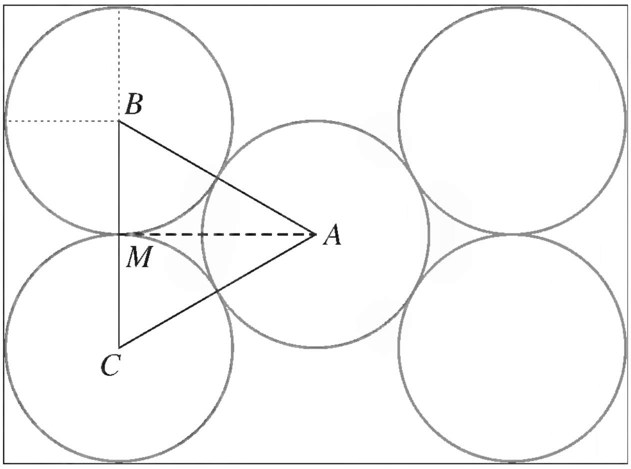
1. Five coins with radius 1 are arranged to touch as shown in the diagram so that the figure has two axes of symmetry. They just fit onto the base of a rectangular box. What is the area of the base?



ANSWER $8(1+\sqrt{3})$

SOLUTION

We join three of the circle centres to make a triangle $\triangle ABC$, and draw the median AM. This is a median by the symmetry of the diagram, and AM is also perpendicular to BC by symmetry.



We have $AB = 2$ and $BM = 1$, so applying Pythagoras's Theorem to $\triangle ABM$ gives

$$\begin{aligned} & AM^2 + BM^2 = AB^2 \\ \text{so}\quad & AM^2 + 1 = 4 \\ \text{thus}\quad & AM = \sqrt{3}. \end{aligned}$$

Noting that B is a distance of 1 from the nearest two edges of the rectangle (as indicated by the two dotted radii in the diagram), we see that A is a distance of $1+\sqrt{3}$ from the left of the rectangle and 2 from the top. Since A is in the centre of the rectangle, the horizontal base of the rectangle is $2(1+\sqrt{3})$ and the height is 4, giving an area of $8(1+\sqrt{3})$.

2. A fighter pilot needs a special seat in order to eject from his aircraft. The rockets attached to the ejector seat must take the seat and pilot sufficiently high to clear the tail fin. Assume that the tail fin sticks up 2 m high but is 20 m behind the cockpit, that the fighter has slowed down to 450 km/h, and that the sudden blast of air stops the pilot moving forward the moment the seat is out of the cockpit.

Find the minimum acceleration (in $\mathrm{m/s^2}$) that the rockets in the ejector seat will have to provide to throw the pilot clear of the tail fin so that it does not hit the pilot.

What is the effect on the answer if the pilot and seat continue to move forward enough to double the time that they have to clear the tail-fin?

Note: $\text{average speed} = \dfrac{\text{distance travelled}}{\text{time taken}}$; $\text{average acceleration} = \dfrac{\text{change of speed}}{\text{time taken}}$

For example, if the change of speed is 10 m/s over 5 s, the average acceleration is $2\,\mathrm{m/s^2}$.

ANSWER $156\,\mathrm{m/s^2}$ to 3 significant figures

SOLUTION

We must first work out the time it takes for the tail-fin to move to the point where the cockpit was when the pilot ejected; call this T s. This requires us first to convert the speed 450 km/h into m/s.

$$450\,\mathrm{km/h} = \frac{450\,\mathrm{km}}{1\,\mathrm{h}} = \frac{450\,000\,\mathrm{m}}{3600\,\mathrm{s}} = 125\,\mathrm{m/s}$$

We can now calculate T, by rearranging the formula for average speed:

$$T\,\mathrm{s} = \frac{20\,\mathrm{m}}{125\,\mathrm{m/s}} = 0.16\,\mathrm{s}.$$

We can now consider the vertical motion, where the pilot needs to move 2 m vertically in T s:

$$\text{average vertical speed} = \frac{2\,\mathrm{m}}{T\,\mathrm{s}} = 12.5\,\mathrm{m/s}$$

Assuming that the vertical speed increases from 0 m/s to a final speed, with the average being 12.5 m/s, we need the final speed to be $2 \times 12.5\,\mathrm{m/s} = 25\,\mathrm{m/s}$. Therefore

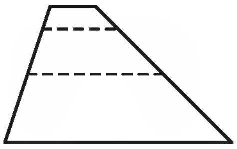
$$\text{average acceleration} = \frac{25\,\mathrm{m/s}}{0.16\,\mathrm{s}} = 156\,\mathrm{m/s^2} \quad \text{(to 3 s.f.)}$$

If the time is doubled, then since we have divided by T twice in the calculations (once to find the average speed and once to find the average acceleration), the average acceleration will be divided by 4, giving $39\,\mathrm{m/s^2}$, which is much better for the ejecting pilot.

> An acceleration of $1g = 9.8\,\mathrm{m/s^2}$ is equivalent to normal gravity. The answer of $156\,\mathrm{m/s^2} \approx 16g$ is on the brink of being survivable, even for such a brief period of time (a fraction of a second), though it is very sudden. As a comparison, an astronaut experiences much less than this, even on re-entry to the atmosphere: the space shuttle crews experienced $3g$ while the Apollo moon mission astronauts underwent $7g$ on re-entry. People on roller-coasters might experience a maximum of $5g$. Fighter pilots in their special suits can work at up to $9g$.

3. A trapezium has parallel sides of lengths 1 and 8. The other two sides have lengths $10\frac{1}{2}$ and 14. Two further lines cut the outer trapezium into three similar trapeziums as shown in the diagram, resulting in four parallel lines. Find the perimeter of the smallest trapezium.

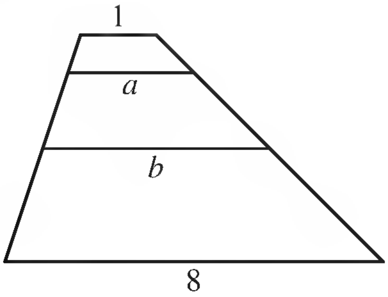
The diagram is not to scale.



ANSWER $6\frac{1}{2}$

SOLUTION

We label the parallel side lengths:



Since the three trapezia are similar, the ratio of the upper side length to the lower side length of each is equal, so

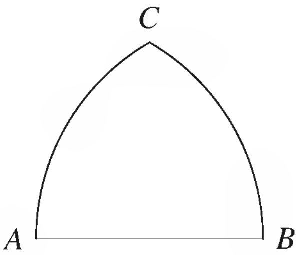
$$\frac{1}{a} = \frac{a}{b} = \frac{b}{8}.$$

The first equality gives $b = a^2$ (on multiplying both sides by ab) and the second gives $8a = b^2$ (multiplying by $8b$).

To solve these equations, we can substitute $b = a^2$ into the second equation to give $8a = (a^2)^2$, so $8a = a^4$. Dividing by a gives $a^3 = 8$, so $a = 2$ and $b = 4$.

We therefore see that the ratio of side lengths of the three trapezia is 1 : 2 : 4, and so the other two sides of the large trapezium are divided in this ratio. Noting that $1 + 2 + 4 = 7$, and $10\frac{1}{2} = 7 \times 1\frac{1}{2}$, we see that the other two sides of the smallest trapezium are $1\frac{1}{2}$ and 2. Therefore its perimeter is $1 + 2 + 1\frac{1}{2} + 2 = 6\frac{1}{2}$.

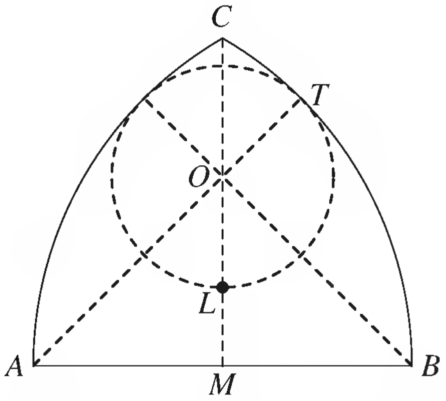
4. A pointed arch ABC (like a Gothic arch) has a base AB of length 2. The curved sections are arcs of circles constructed with centres A and B, each with radius 2, meeting at C. A circle is inscribed in the arch so that it touches the base and each of the arcs. Find the radius of this circle.



ANSWER $\frac{3}{4}$

SOLUTION

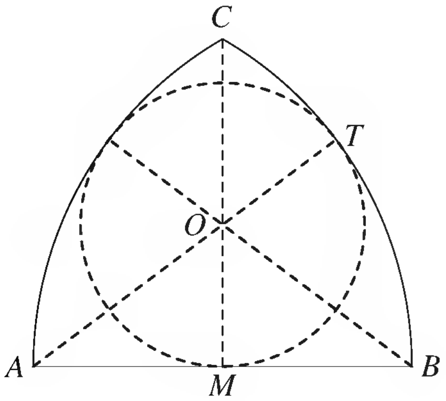
We start by thinking about how to draw such a circle. Trying to make it touch all three sides simultaneously is harder than just having it touch two, so we first consider how we could draw a circle which touches the two curved sides. If we draw a circle with its centre lying on the line of symmetry of the figure, we can adjust its radius to touch these two sides. Thus let M be the midpoint of AB, and draw the line of symmetry CM. Let O be a point on CM, and draw radii from A and B through O. We can then draw a circle with centre O touching the two curved sides at the ends of the radii. We have labelled one of these points as T in the following diagram.[1]



This circle crosses the line OM at L, and in order to achieve our objective, we need L to be at M. So let us sketch the diagram with O a little lower so that L lies at M:

---

[1] A detailed justification that this circle does indeed touch the two curved sides and not cross them is somewhat subtle, and would not be expected at this level. One way to do so is as follows. Let U be some other point on the arc BC, and assume that U also lies on the circle with centre O. Now consider the triangle AOU. Note that AU is a radius of the circle centred at A, so $AU = AT$. As U is supposed to lie on the circle centred at O, $OU = OT$. Thus $AO + OU = AO + OT = AT = AU$, and so the triangle is degenerate. The only way this can happen is if O lies on the line AU, which occurs precisely when U is at T itself.



The radius of the inscribed circle is $r = OM = OT$. Then $AO = 2 - r$. By Pythagoras's theorem, we therefore have

$$\begin{aligned} & OM^2 + AM^2 = AO^2 \\ \text{so} \quad & r^2 + 1^2 = (2-r)^2. \\ \text{Expanding gives:} \quad & r^2 + 1 = 4 - 4r + r^2 \\ \text{so} \quad & 4r = 3 \\ \text{giving} \quad & r = \tfrac{3}{4}. \end{aligned}$$

5. Find all the real solutions of the equation $x = \left|2x - |60 - 2x|\right|$.

Note that $|x|$ means the *absolute value* or *modulus* of x, which is its distance from zero. So $|4| = 4$ while $|-3| = 3$. More formally, we can say that $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. The graph of $y = |x|$ is a V-shape.

ANSWER $x = 12$, 20 or 60

SOLUTION

We solve this equation by splitting into cases, as suggested in the hint.

Case (a): Assume $x \leq 30$. Then $|60 - 2x| = 60 - 2x$, so the equation can be written $x = |2x - (60 - 2x)|$, which simplifies to $x = |4x - 60|$. There are now two sub-cases, depending on whether $4x - 60$ is positive or negative.

(i) If $x \geq 15$, then $4x - 60 \geq 0$, so the equation becomes $x = 4x - 60$. Hence $3x = 60$, giving $x = 20$. This is consistent with both $x \geq 15$ and $x \leq 30$, so this is a valid solution (which can also be confirmed by substituting it back into the original equation).

(ii) If $x < 15$, then $4x - 60 < 0$, so the equation becomes $x = -(4x - 60)$. This simplifies to $5x = 60$, giving $x = 12$. This is consistent with both $x < 15$ and $x \leq 30$, so this is another valid solution.

Case (b): Assume $x > 30$. Then $|60 - 2x| = -(60 - 2x) = 2x - 60$, so the equation can be written $x = |2x - (2x - 60)|$, which simplifies to $x = |60| = 60$. This is consistent with $x > 30$, so this is also a valid solution.

We have therefore found three valid solutions to the equation.

6. A cuboid with a square base is inscribed in a sphere. (This means that the vertices of the cuboid all lie on the surface of the sphere.) The height of the cuboid is four times the length of a base edge. Find the ratio of the surface area of the cuboid to the surface area of the sphere.

The surface area of a sphere with radius r is $4\pi r^2$.

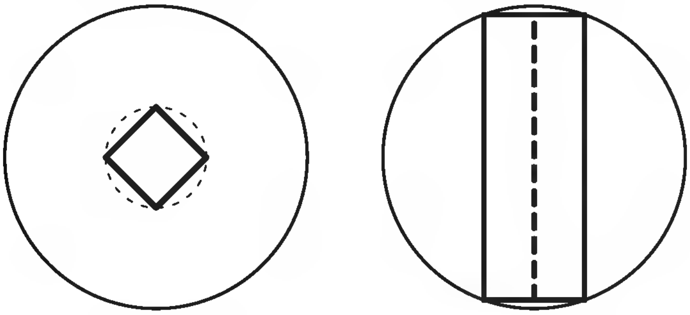
ANSWER $1 : \pi$

SOLUTION

We present two different approachs to this question. The first uses an ingenious way of looking at the cuboid, while the second is more straightforward in setting up the problem, but requires more geometry and algebra to solve it.

Approach 1

For this approach, we view the cuboid from above and also from the side, looking directly towards a vertical edge, which is shown dashed in the right-hand diagram.



Let a be the length of an edge of the square base. By applying Pythagoras's theorem to the square, we calculate that the diagonal of the square has length $a\sqrt{2}$. (We write $a\sqrt{2}$ rather than $\sqrt{2}a$ so that it is clear exactly what is being square-rooted.) This diagonal is what you see across the top and bottom of the right-hand diagram.

Now consider the rectangle in the right-hand diagram. The top side is $a\sqrt{2}$ and the vertical is $4a$ (as we are told that the height is four times the square side length). Let the radius of the sphere be r. We can apply Pythagoras's theorem to the diagonal of this rectangle, which has length $2r$:

$$\begin{aligned} & & (a\sqrt{2})^2 + (4a)^2 &= (2r)^2 \\ &\text{so} & 2a^2 + 16a^2 &= 4r^2 \\ &\text{therefore} & \tfrac{9}{2}a^2 &= r^2. \end{aligned}$$

The surface area of the cuboid is $2 \times a^2 + 4 \times a \times 4a = 18a^2$.

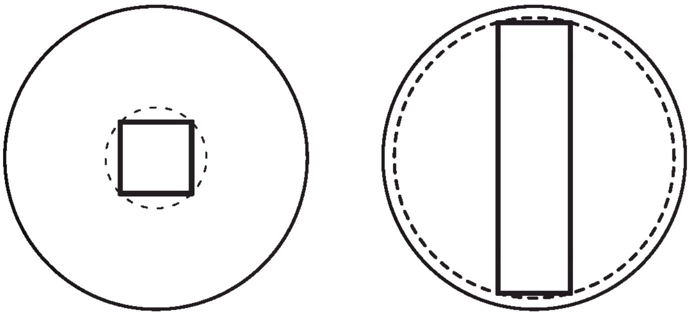
The surface area of the sphere is $4\pi r^2 = 4\pi \times \frac{9}{2}a^2 = 18\pi a^2$.

Thus we see that the ratio of the surface area of the cuboid to the surface area of the sphere is $1 : \pi$.

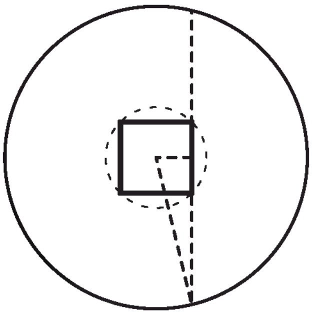
Notice that we did not have to take any square roots as all the area formulae use a^2 or r^2.

Approach 2

This time, when we look at the cuboid from the side, we look directly at a face. We see the following:



The right-hand diagram requires some explanation. We are looking at a face of the cuboid from the side. But this face does not pass through the centre of the sphere, so the top and bottom edges do not reach the edge of the sphere seen in the side projection. Instead, they lie on the circumference of a smaller circle. We can work out the radius of the smaller circle using the left-hand diagram. We redraw it as follows:



The vertical dashed line is the diameter of the smaller circle. We let:

r = radius of sphere

R = radius of smaller circle

$2b$ = side length of square base of cuboid

(Note that we use $2b$ rather than b for simplicity, as we are going to be using half this length in the following calculations.) The dashed triangle has side lengths b, R and r, so by Pythagoras's theorem,

$$r^2 = b^2 + R^2.$$

Now looking at the right-hand diagram above, the rectangle has side lengths $2b$ and $8b$, and the diagonal has length $2R$, so by Pythagoras's theorem again, we have

$$(2b)^2 + (8b)^2 = (2R)^2$$

which simplifies to $R^2 = 17b^2$. Thus $r^2 = b^2 + R^2 = 18b^2$.

The surface area of the cuboid is $2(2b)^2 + 4(2b)(8b) = 72b^2$, and the surface area of the sphere is $4\pi r^2 = 72\pi b^2$. Therefore the ratio of the surface area of the cuboid to the surface area of the sphere is $1 : \pi$.

7. One fifth of students in a form group of 30 have blue eyes. Ben is one of the students with blue eyes, but no blue-eyed students were chosen by their new tutor Ms Loveplay to take one of the 6 parts in that year's inter-form drama competition. He feels upset because he has always been good at acting.

On a different occasion, Ben attended a masterclass where they worked out the probability of 12 randomly chosen students each having a different birthday. Ignoring leap years, it is

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \frac{355}{365} \times \frac{354}{365} = 83.3\% \text{ (to 1 decimal place).}$$

The masterclass students were surprised that the probability of two or more of them having the same birthday was about the same as that of throwing a six with a standard die.

Based on this idea, he decides to make a case to put to Ms Loveplay that she is discriminating against blue-eyed students. If the actors had been chosen randomly from the form, what is the probability that none of the actors had blue eyes? Should he complain?

ANSWER about 22.7%; no

SOLUTION

The probability that the first actor chosen does not have blue eyes is $\frac{24}{30}$. Assuming that happened, the probability that the second actor chosen does not have blue eyes is $\frac{23}{29}$, and so on up to the sixth actor. The overall probability that none of the actors had blue eyes is therefore

$$\frac{24}{30} \times \frac{23}{29} \times \frac{22}{28} \times \frac{21}{27} \times \frac{20}{26} \times \frac{19}{25} = 0.227 \quad \text{(to 3 decimal places)}$$

This is quite a high probability, almost a quarter. Though Ben could of course complain, he does not have reasonable grounds to do so just on the basis of it being an unlikely occurrence.

> He would be almost certain to have a case if the problem had been about a choosing a football team from the class. For 11 players, the probability of none of them being blue-eyed, if they are chosen at random, is about 4.6%. Values below 5% are often taken as being *significant*. This means that the chance is small enough to suggest that the cause may not be random.

8. In the following puzzle each different capital letter represents a different digit. Thus $SEVEN_{10}$ represents a five digit decimal number.

$SEVEN_{10}$ is a prime and $SEVEN_{10}$ minus $THREE_{10}$ equals $FOUR_{10}$. Curiously, though, $FOUR_{10}$ *is* a prime (as is $RUOF_{10}$) but $THREE_{10}$ is *not* prime. Another oddity is that TEN_{10} is a perfect square. Find the values of $FOUR_{10}$ and TEN_{10}.

ANSWER $FOUR_{10} = 3407$ and $TEN_{10} = 529$

SOLUTION

We note that all ten digits 0 to 9 are needed (the ten letters are E, F, H, N, O, R, S, T, U, V). We will drop the suffix $_{10}$ in what follows.

Since $THREE + FOUR = SEVEN$ and S and T are different, $S = 1 + T$ (as a one must have been carried from the thousands to ten-thousands column). Hence T cannot be 9 and S cannot be 0.

Since $SEVEN$, $FOUR$ and $RUOF$ are primes, N, F and R must use three of the digits 1, 3, 7 and 9 (as every prime with more than one digit ends with one of these. Because N is also the units digit of the square TEN, N can only be 1 or 9. Thus TEN can only be one of the squares 169, 289, 361, 529, 729, 841 or 961, of which we exclude 961 as we know that $T \neq 9$.

Next consider the ones (units) and tens digits of $THREE + FOUR = SEVEN$. R and U can be deduced from knowing E and N; in particular, U can only be 0 or 9.

We tabulate our current possibilities for TEN and the resulting values of $S = T + 1$, R and U; ignore case (e) and the remaining five columns for a moment.

Case	T	E	N	S	R	U	F	H	V	O	works?
(a)	1	6	9	2	3	0	7	8 (carry)	4 or 5	5 or 4	no
(b)	2	8	9	3	1	0	7	–			
(c)	3	6	1	4	5	9					
(d)	5	2	9	6	7	0	1	–			
(e)							3	8 (carry)	1 or 4	4 or 1	yes ($O = 4$)
(f)	7	2	9	8	7	0					
(g)	8	4	1	9	7	9					

In three cases, we proceed no further because either R is an invalid digit (case (c)) or a digit is used more than once (cases (f) and (g)).

The next digit we can determine is F, which is one of 1, 3, 7 or 9 as noted above. Excluding those digits which have already been used, we are left with $F = 7$ in cases (a) and (b), and $F = 1$ or $F = 3$ in case (d), which we therefore split into two cases, (d) and (e).

From the thousands place in $THREE + FOUR = SEVEN$, we have $H + F = E$ or $H + F + 1 = E$, depending on whether there is a carry from $R + O$ in the hundreds place; filling in the possibilities for H then eliminates cases (b) and (d), and also gives the possible values for the remaining digits.

In case (a), this does not produce the required carry for $R + O$, so this case is not possible. Finally, in case (e), $R + O$ gives a carry if we choose $V = 1$, $O = 4$. We can now confirm that $58722 + 3407 = 62129$ works.

It is straightforward (though tedious) to check that 62129, 3407 and 7043 are prime. Clearly $THREE = 58722$ is not prime since it is even.

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Archimedes

Sheet 5

Solutions and comments

This programme of the Mentoring Scheme is named after Archimedes of Syracuse (287–212 BCE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Archimedes.html for more information.

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University of Leeds, Leeds LS2 9JT

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Version 1.1, Feb 2020

1. The hypotenuse of a right angled triangle has length c and one of the other sides has length b. Furthermore, b and c are consecutive integers. Prove that the square of the length of the third side is $b + c$.

SOLUTION

We can write $c = b + 1$ or $b = c - 1$. Let a be the length of the third side. Then by Pythagoras's theorem, we have $a^2 + b^2 = c^2$, or $a^2 = c^2 - b^2$. Now substituting in our expression for c, we obtain

$$\begin{aligned} a^2 &= c^2 - b^2 \\ &= (b+1)^2 - b^2 \\ &= b^2 + 2b + 1 - b^2 \\ &= 2b + 1 \\ &= b + b + 1 \\ &= b + c. \end{aligned}$$

Alternatively, we can use the difference of two squares to give

$$\begin{aligned} a^2 &= c^2 - b^2 \\ &= (c-b)(c+b) \\ &= 1 \times (c+b) \\ &= b + c. \end{aligned}$$

> You may well be familiar with Pythagorean triples such as $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$ and so on. These are possible values for (a, b, c). In general, the possible values of (a, b, c) are $((2k+1)^2, 2k^2 + 2k, 2k^2 + 2k + 1)$ for a positive integer k.

2. Take 8 cards labelled 1 to 8. Holding the cards face up, deal them into four piles, starting from the left: deal one card onto each position in turn, until there are two cards in each pile. Pick up the four piles in order, starting with the left pile on top still face up. Repeat this process (dealing, then picking them up) until all the cards are back in their original order.

How many times do you deal out the cards, and which cards return more than once to their original position during this process?

ANSWER 6 deals; the 3rd and 6th cards return to their original positions more than once

SOLUTION

We can express the effect of one deal by writing down the order of the cards before and after a deal: if the cards start in the order $(1, 2, 3, 4, 5, 6, 7, 8)$, then they end in the order $(5, 1, 6, 2, 7, 3, 8, 4)$. We can also show how the *positions* of the cards change:

original position	1	2	3	4	5	6	7	8
final position	2	4	6	8	1	3	5	7

The positions cycle as follows: $1 \to 2 \to 4 \to 8 \to 7 \to 5 \to 1$ and $3 \to 6 \to 3$. The cards $(1, 2, 4, 8, 7, 5)$ cycle back to their starting point after every six deals while the cards $(3, 6)$ cycle back after every two deals. It follows that all eight cards are back in their original positions after 6 deals.

> You might like to observe in relation to question 6 that if we consider multiplication of remainders on division by 9 (rather than by 7 as in that question), the final position is the original position times 2. Why is this the case?

3. Alex is still quite young and has to take 5 steps to cover the same distance that his father walks in 3 steps. Alex takes 7 steps in the time that his father takes 6 steps. Alex sets off along a footpath ahead of his father and takes 30 steps before his father sets off behind him.

How many steps does the father take to catch up with his son?

ANSWER 60 steps

SOLUTION

Approach 1

This approach does not involve algebra. Let Alex walk a distance of 1 unit in 5 steps. This is the distance his father walks in 3 steps. In the time his father walks 6 steps, Alex walks 7 steps. Thus his father walks 2 units in the time that Alex walks $\frac{7}{5}$ units.

Hence every 6 steps that his father takes, he catches Alex up by $2 - \frac{7}{5} = \frac{3}{5}$ unit. Since Alex starts $30 \div 5 = 6$ units ahead, his father takes $6 \div \frac{3}{5} = 10$ sets of 6 steps to catch up, that is 60 steps.

Approach 2

You can avoid fractions in the algebra by choosing a unit x of distance so that Alex's step length is $3x$ and his father's step length is $5x$. Assume that his father takes $6n$ (larger) steps while Alex takes a further $7n$ (smaller) steps until the father catches up.

Comparing total distances walked we get the equation:

$$\begin{array}{lrcl} & 6n \times 5x &=& (30 + 7n) \times 3x \\ \text{dividing by } x \text{ gives} & 6n \times 5 &=& (30 + 7n) \times 3 \\ \text{so simplifying,} & 30n &=& 90 + 21n \\ \text{hence} & n &=& 10. \end{array}$$

Therefore the father takes $6n = 60$ steps to catch up Alex.

> You will notice that the lengths of the steps do not need to be known.

4. The Fibonnacci sequence is defined by the relation:

$$F_{n+1} = F_n + F_{n-1},$$

where F_n means the current (nth) term, F_{n-1} means the previous (($n-1$)th) term and F_{n+1} means the next (($n+1$)th) term. We start the sequence with $F_1 = 1$ and $F_2 = 1$. (It is also convenient to let $F_0 = 0$.) Write out the values from F_1 to F_{10}.

Prove that in general

$$F_{n+5} = 8F_n + 5F_{n-1}.$$

(F_{n+5} means the term 5 places on from the current term, that is, the $(n+5)$th term.)

Answer 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

Solution

Each term is the sum of the previous two terms, so we obtain the sequence given.

To find an expression for F_{n+5}, we can work our way up:

$$\begin{aligned}F_{n+2} &= F_{n+1} + F_n\\ &= (F_n + F_{n-1}) + F_n\\ &= 2F_n + 1F_{n-1}.\\ F_{n+3} &= F_{n+2} + F_{n+1}\\ &= (2F_n + 1F_{n-1}) + (F_n + F_{n-1})\\ &= 3F_n + 2F_{n-1}.\end{aligned}$$

We can continue this process twice more to obtain the required result. On the other hand, we could use $F_{n+3} = 3F_n + 2F_{n-1}$ to write $F_{n+5} = 3F_{n+2} + 2F_{n+1}$. Then we substitute the relation above and the definition of F_{n+1} to achieve the final answer:

$$\begin{aligned}F_{n+5} &= 3(2F_n + 1F_{n-1}) + 2(F_n + F_{n-1})\\ &= 8F_n + 5F_{n-1}.\end{aligned}$$

The relation to be proved can also be written:

$$F_{n+5} = F_6F_n + F_5F_{n-1},$$

which allows one to investigate further such relationships. For example, is it true that

$$F_{n+6} = F_7F_n + F_6F_{n-1}$$

or, even more generally, that

$$F_{n+m} = F_{m+1}F_n + F_mF_{n-1}?$$

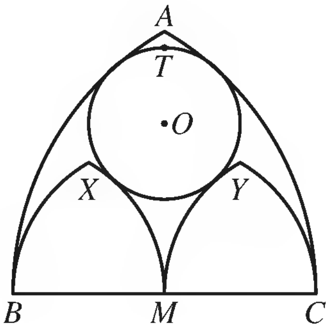
What about other variations? For example, we could define a sequence L_n with $L_{n+1} = L_n + L_{n-1}$, the same rule as before, but with the starting conditions $L_1 = 1$

and $L_2 = 3$. This gives the sequence beginning 1, 3, 4, 7, 11, 18, called the Lucas sequence. Does this still have the property $L_{n+5} = 8L_n + 5L_{n-1}$ or would it now be $L_{n+5} = 18L_n + 11L_{n-1}$? Or is there no such relationship?

5. The diagram shows the outline of the principal roundel and two arches in the great east window of Lincoln Cathedral. The arches are formed from arcs of circles, their centres being at either B, C or M, which is the midpoint of BC. Suppose that $BM = 2$. Note that $\triangle ABC$, $\triangle XBM$ and $\triangle YCM$ are all equilateral.

Find the diameter of the circle with centre O touching the arches as shown in the diagram.

If T is where AM cuts the top of the circle with centre O, find the ratio $BM : TM$. Leave square roots in your answer.



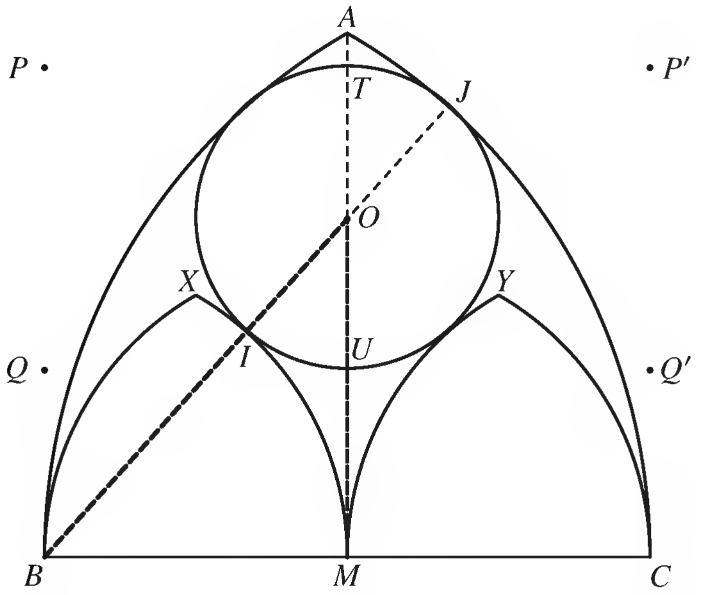
You can ask your mentor for more information about the window itself if you are interested, and there are photographs available on the internet.

Answer Diameter = 2, $BM : TM = 1 : \frac{1}{2}(1 + \sqrt{5})$

Solution

The line joining the centres of two touching circles passes through the point where they touch, as it is perpendicular to their common tangent. So if we draw the line BO extended, it will pass through the point where the arc MX and the circle centred at O meet, and also through the point where the arc AC and the circle centred at O meet. We also draw the line of symmetry AM.

(Ignore the points labelled P, Q, P_1 and Q_1 for now; these are discussed in the commentary below.)



The diameter of the circle centred on O is the length of the line IJ. But $BJ = 4$ as it is a radius of the arc AC, and $BI = 2$ as it is a radius of the arc MX. So $IJ = 2$.

We now aim to calculate $TM = TO + OM$. We have $TO = 1$, as it is a radius of the circle centred on O. Now consider $\triangle OBM$. $OB = 3$ and $BO = BI + IO = 3$. By Pythagoras's Theorem, we can deduce that $OM^2 = BO^2 - BM^2 = 3^2 - 2^2 = 5$, so $OM = \sqrt{5}$. Therefore $BM : TM = 2 : 1 + \sqrt{5}$, which can be rewritten as $BM : TM = 1 : \frac{1}{2}(1 + \sqrt{5})$.

> This is the *golden ratio*. If the lines TM and MB are used as two sides of a rectangle with fourth vertex at P, as shown in the diagram, we have a *golden rectangle*. We note that if a rectangle is formed with two sides UM and BM and fourth vertex at Q, then $PTUQ$ is a square. The rectangle that is left, $QUMB$, has ratio of sides
>
> $$UM : BM = \sqrt{5} - 1 : 2.$$
>
> It is a useful exercise to show that this ratio is equal to the ratio $BM : TM$, which shows that $QUMB$ is also a golden rectangle. In fact, one way of defining a golden rectangle is as a rectangle for which removal of a square in this way leaves behind a similar rectangle.
>
> We can introduce points P' and Q' as the reflections of P and Q in AM. Then $BMTP$ can be rotated 90° clockwise about M and enlarged by factor $\frac{1}{2}(\sqrt{5} - 1)$ with centre M to map onto $UMCQ'$. (It is a little messier to map directly onto $UMBQ$.)

6. (a) When the integer a is divided by 7, the remainder is 3. When the integer b is divided by 7, the remainder is 4. Show that when ab is divided by 7, the remainder is 5.

(b) This remainder multiplication table shows the remainder when two numbers are multiplied and the result is divided by 7. For example, the result from part (a) is shown in the row labelled 3 and column labelled 4. Complete the table.

	1	2	3	4	5	6
1	.	.	.	.	.	.
2	.	.	.	.	.	.
3	.	.	.	5	.	.
4	.	.	.	.	.	.
5	.	.	.	.	.	.
6	.	.	.	.	.	.

(c) Can you explain why 0 does not appear in the table? Furthermore (and this is more difficult), why does each value from 1 to 6 appear in each row and each column once and once only?

Solution

(a) We can write $a = 7m + 3$, for some integer m. Similarly, $b = 7n + 4$ for some integer n. (We need to use a different letter here, as m and n might not be equal.) Then

$$\begin{aligned} ab &= (7m+3)(7n+4) \\ &= 49mn + 28m + 21n + 12 \\ &= 7(7mn + 4m + 3n) + 12 \\ &= 7(7mn + 4m + 3n + 1) + 5 \end{aligned}$$

so the remainder on dividing ab by 7 is 5.

(b) A similar argument to part (a) works regardless of what the remainders are: if a leaves remainder k and b leaves remainder l, then ab leaves the same remainder as kl does. So to complete the table, we might as well just multiply the numbers 1 to 6.

We can also note that since $ab = ba$ for every pair of numbers a and b (multiplication is *commutative*), the table is symmetric about the diagonal from top left to bottom right. This means that we only need to calculate about half of the table.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(c) A remainder of 0 is the same as being divisible by 7. So if 0 were to appear, then the product of the two numbers must be divisible by 7. A key property of prime numbers is the following:

If p is a prime number, and p divides ab, then p divides a or p divides b.

(In more advanced mathematics, this is often taken to be the definition of a prime number; the property that the only factors of p are 1 and itself follows from this property, as if $p = ab$, then p must divide a or b; if both are less than p, this is impossible.)

Because 7 is prime, if the product of two numbers is a multiple of 7, then one of the two numbers must be a multiple of 7. Since no number leaving a remainder of 1 to 6 is a multiple of 7, no product of two such numbers can be. Therefore no 0 appears in the table.

(Note that this is not the case if we used a non-prime number to divide by. For example, if we look at remainders on dividing by 12, 2×6 leaves a remainder of 0, even though neither 2 nor 6 leaves a remainder of 0 itself.)

Let us now explain why the same number cannot appear twice in a row. Once we have done this, then by the symmetry of the table, this will also show that no number can appear twice in a column. Furthermore, since there are 6 numbers from 1 to 6 and six columns in each row, each of the numbers will therefore appear exactly once in each row and exactly once in each column.

So let us suppose that the same number did appear twice in a row. Let us say that $2 \times a$ and $2 \times b$ leave the same remainder on division by 7, where a and b are different numbers between 1 and 6. Then we can write

$$\begin{aligned} 2a &= 7k + r \\ 2b &= 7m + r \end{aligned}$$

where k and m are integers. If we now subtract these, we get

$$2(a - b) = 7(k - m),$$

which is to say that 7 divides $2(a - b)$. But since 7 is prime, this means that 7 divides 2 or 7 divides $(a - b)$. Clearly 7 does not divide 2, so $(a - b)$ must be a multiple of 7. But this is impossible as $a - b$ lies between -5 and 5, and is not zero. So $2 \times a$ and $2 \times b$ must be different.

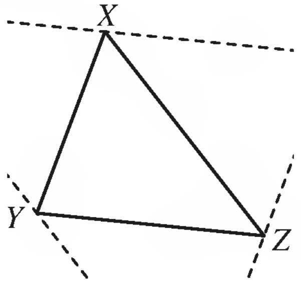
Exactly the same argument works if we replace 2 by any number between 1 and 6, and we are done.

> A table where each value appears once and once only in each row and in each column is called a *Latin square*. Such arrangements are sometimes used in statistical analysis, say when a farmer is testing various treatments on a field with varying fertility across the field. It is also a basic property of a *group* multiplication table in mathematics.

7. Triangle XYZ is acute-angled.

(a) Lines are drawn through X parallel to YZ, through Y parallel to ZX and through Z parallel to XY as shown in the figure. These new lines through Y and Z meet at A, those through Z and X meet at B, and those through X and Y meet at C.

Prove that X is the mid-point of BC.



(b) The line perpendicular to YZ through X is the perpendicular bisector of BC. Show that the perpendicular bisectors of BC, CA and AB meet in one point. Call this point H.

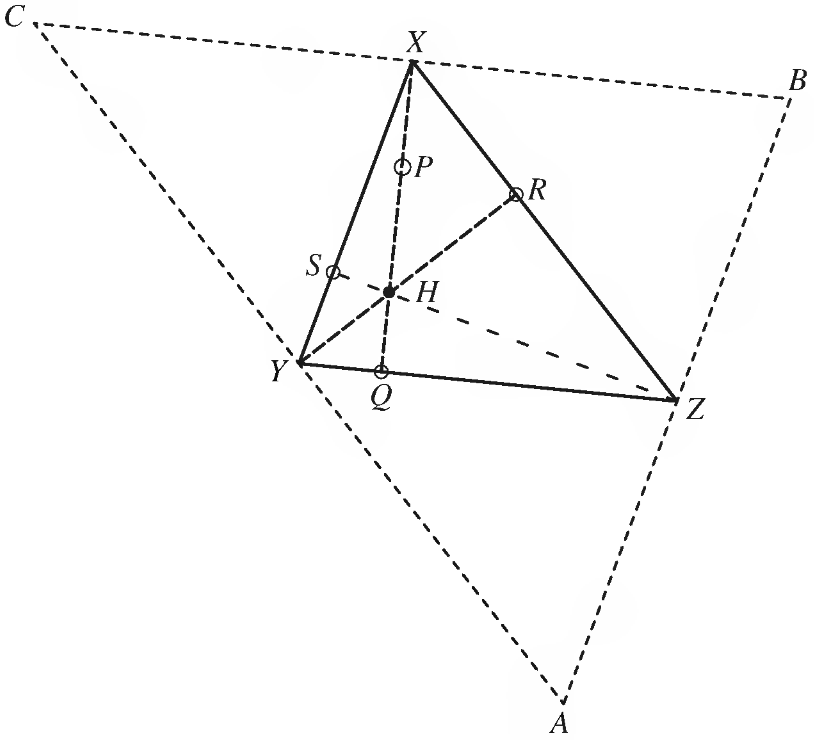
(c) The result of (b) shows that the three altitudes of $\triangle XYZ$ meet at H. (An *altitude* of a triangle is a line through a vertex perpendicular to the opposite side.) Show that

$$\angle HYZ + \angle HZY = \angle YXZ.$$

(d) Do these results still hold if $\angle YXZ$ is obtuse?

Solution

The completed diagram is as follows. (The points P, Q, R and S will be explained below.)



(a) CB is parallel to YZ and BA is parallel to XY. Hence $XYZB$ is a parallelogram, and so $XB = YZ$. By similar reasoning $CXZY$ is a parallelogram, giving $CX = YZ$. Hence $CX = XB$, and X is the midpoint of BC.

(b) This line is perpendicular to BC as BC is parallel to YZ, and the line is perpendicular to YZ. It passes through X, the midpoint of BC, so is the perpendicular bisector of BC.

Now take a point such as P on this perpendicular bisector. A key property of the perpendicular bisector is that P is equidistant (equally distant) from B and C. A corollary of this is that a circle with centre P passing through B would also pass through C. Likewise any point on the perpendicular bisector of CA will be equidistant from C and A. Let H be the point where these two bisectors meet. Because H lies on both perpendicular bisectors, it is equidistant from B, C and A. Hence H also lies on the perpendicular bisector of AB, and so all three perpendicular bisectors meet at this point.

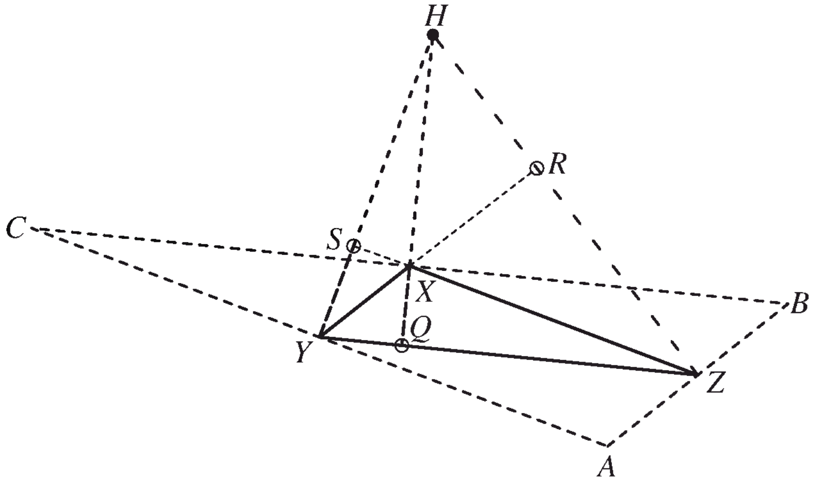
(c) Consider the angles in the triangle AYZ. We know that $\angle AYH = \angle AZH = 90°$. We also know that $\angle YAZ = \angle YXZ$ as $AYXZ$ is a parallelogram. We then have, using the sum of angles in the triangle AYZ,

$$\begin{aligned} & \angle AYZ + \angle AZY + \angle YAZ = 180° \\ \text{so} \quad & (90° - \angle HYZ) + (90° - \angle HZY) + \angle YXZ = 180° \\ \text{giving} \quad & 180° + \angle YXZ = 180° + \angle HYZ + \angle HZY \\ \text{hence} \quad & \angle YXZ = \angle HYZ + \angle HZY. \end{aligned}$$

Alternatively we can label the feet of the perpendiculars from X, Y and Z as Q, R and S, as shown in the diagram. Then $\triangle XQY$ is similar to $\triangle ZSY$ because both triangles are right-angled and share an angle at Y. Hence $\angle HZY = \angle YXQ$. Likewise $\angle HYZ = \angle ZXQ$.

Finally $\angle YXQ + \angle ZXQ = \angle YXZ$, so the result is proved.

(d) The relations and proofs still hold.



In this case, H is outside the triangle XYZ, as shown. The proofs of the earlier parts still work perfectly well, though now the perpendicular to ZX from Y meets ZX beyond X at S, and then itself has to be extended upwards until it meets the perpendicular from X which is also extended.

> The point H is known as the *orthocentre* of $\triangle XYZ$. It is also the *circumcentre* of $\triangle ABC$, which is the centre of the circle passing through all three of A, B and C.
>
> The observant student may notice that $\triangle HYZ$ in the final diagram is the same as $\triangle XYZ$ in the first diagram except for the labelling: the roles of X and H have been switched and the positions of A, B and C have been moved in order to accommodate this. The circumcentre of $\triangle ABC$ is now outside $\triangle ABC$. This switch makes it relatively easy to redraw the diagram. This gives the rather neat result that if H is the orthocentre of $\triangle XYZ$, then X is the orthocentre of $\triangle HYZ$, Y is the orthocentre of $\triangle XHZ$, and Z is the orthocentre of $\triangle XYH$.

8. Let a and b be two positive integers that have no common factor greater than 1. (Such numbers are called *coprime*.)

In this question, we will call a positive integer *composable* if it can be produced by adding a's and b's, using each as many times as required (or not at all if need be).

For example, with $a = 5$ and $b = 7$, 26 is composable because $26 = 5 + 7 + 7 + 7$, but 23 is not. Other examples of composable numbers with $a = 5$ and $b = 7$ are 5, 7, 21 and 30.

(a) In the case $a = 5$ and $b = 7$, how many positive integers less than 35 are *not* composable?

(b) Now let a and b be any two positive coprime integers. Show that there are $\frac{1}{2}(a-1)(b-1)$ positive integers less than ab that are not composable.

(c) In the case $a = 5$ and $b = 7$, show that every integer greater than or equal to 35 is composable.

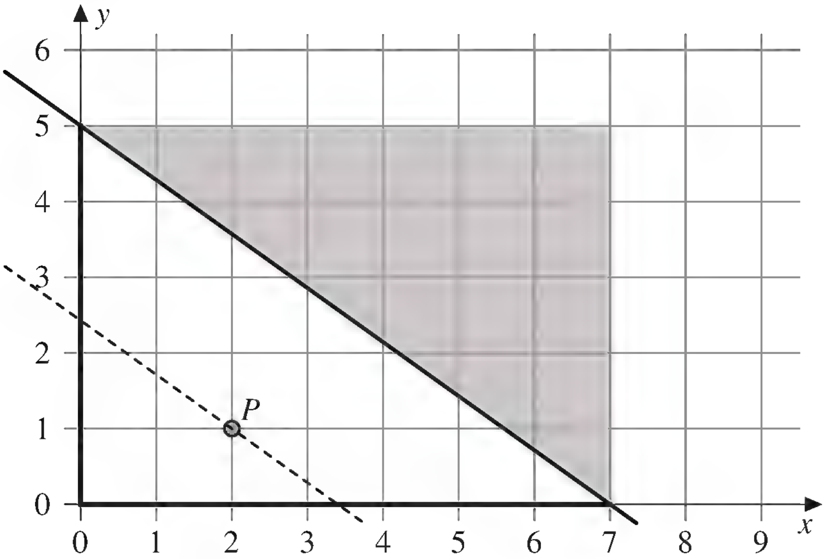
(d) Bonus question (hard!): Show that for any two positive coprime integers a and b, every integer greater than or equal to ab is composable.

ANSWER (a) 12

SOLUTION

(a) For the n that we are testing to be composable, we require non-negative integer values of x and y so that $5x + 7y = n$. Note that if $x \geq 7$ or $y \geq 5$, then $n \geq 35$ which is outside the range that we are investigating.

$5x + 7y = 35$ is a straight line on a graph passing through the points $(7, 0)$ and $(0, 5)$. Lines $5x + 7y = n$ are parallel to this. The figure below shows $5x + 7y = 17$, which passing through the point $P(2, 1)$; note that $5 \times 2 + 7 \times 1 = 17$, so 17 is composable. In this way, each line of the form $5x + 7y = n$ passing through a point in the region bounded by the lines $x = 0$, $y = 0$ and $5x + 7y = 35$ represents the composable number n. It is easier to count the number of points between the lines $x = 0$, $y = 0$, $x = 7$ and $y = 5$. This is the rectangle whose upper half is shaded grey. The upper half is the lower half rotated, so we obtain the total number of composable numbers between 1 and 34 by halving the total.



There are $6 \times 8 = 48$ points within or on the boundary of this rectangle. We must then exclude the 4 corners $(0, 0)$, $(7, 0)$, $(0, 5)$ and $(7, 5)$, as these correspond to $n = 0$ or $n = 35$ (and $(7, 5)$ is the rotated version of $(0, 0)$). We are thus left with 44 points. There are thus 22 composable numbers less than 35. Since we are only looking at $1 \leq n \leq 34$, there will be $34 - 22 = 12$ non-composable numbers in this range.

(b) Using exactly the same argument as in the previous part, we find that there are $(a+1)(b+1)-4$ composable points in the rectangle. We halve this and subtract from $ab - 1$ (since we are counting the number of non-composable numbers less than ab). We then simplify:

$$\begin{aligned}(ab - 1) - \tfrac{1}{2} \times ((a + 1)(b + 1) - 4) &= ab - 1 - \tfrac{1}{2}(ab + a + b + 1 - 4)\\ &= ab - 1 - \tfrac{1}{2}ab - \tfrac{1}{2}a - \tfrac{1}{2}b + \tfrac{3}{2}\\ &= \tfrac{1}{2}ab - \tfrac{1}{2}a - \tfrac{1}{2}b + \tfrac{1}{2}\\ &= \tfrac{1}{2}(a - 1)(b - 1).\end{aligned}$$

Alternatively, we can expand the given expression $\frac{1}{2}(a - 1)(b - 1)$ and show that this is equal to $\frac{1}{2}ab - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}$.

(c) As suggested in the hint, we fix the number of 7's and write down the sequence of composable numbers using this number of 7's, that is, the values of $5x + 7y$ for $x = 0, 1, 2, \ldots$.

number of 7's	composable numbers
0	(0), 5, 10, 15, 20, 25, 30, 35, 40, ...
1	7, 12, 17, 22, 27, 32, 37, 42, 47, ...
2	14, 19, 24, 29, 34, 39, 44, 49, 54, ...
3	21, 26, 31, 36, 41, 46, 51, 56, 61, ...
4	28, 33, 38, 43, 48, 53, 58, 63, 68, ...

The rows are distinguishable by the remainders the numbers leave when divided by 5. With no 7's, there is no remainder; with one 7, there is a remainder of 2; with two 7's, there is a remainder of 4; with three 7 s, there is a remainder of 1, and with four 7's, there is a remainder of 3.

So every possible remainder appears, and as the sequences increase in steps of 5, almost all positive integers appear. The only consideration is that some of the sequences begin later than others; since the largest multiple of 7 considered at the start of one of these rows is 28, all numbers greater than or equal to 28 appear. This table also shows that $23 = 4 \times 7 - 5$ is the largest non-composable number, being the number which would appear immediately before 28 if we continued the sequence backwards.

(d) Let us make a table of multiples of b and the composable numbers using this number of b's, as in the previous part:

number of b's	composable numbers
0	(0), a, $2a$, $3a$, $4a$, ...
1	b, $a + b$, $2a + b$, $3a + b$, $4a + b$, ...
2	$2b$, $a + 2b$, $2a + 2b$, $3a + 2b$, $4a + 2b$, ...
3	$3b$, $a + 3b$, $2a + 3b$, $3a + 3b$, $4a + 3b$, ...
$\vdots$	$\vdots$
$a - 1$	$(a - 1)b$, $a + (a - 1)b$, $2a + (a - 1)b$, $3a + (a - 1)b$, ...

The argument in the previous part considered the remainder on division by $a = 5$, and found that each row of this table accounted for exactly one possible remainder. In our more general setup, we would like to show that numbers with remainder $0, 1, 2, \ldots, a - 1$ on division by a each appear on one row of this table. The remainders are constant on each row, as we add a to get the next term, which does not change the remainder. There are a rows and a possible remainders, so the only way that this would not give all possible remainders is if two rows had the same remainder.

Let us suppose, then, that rb and sb have the same remainder on division by a, where $0 \leq r < s < a$. Then $sb - rb$ is a multiple of a, that is, $(s - r)b$ is divisible by a. Now a and b are coprime, that is, they have no factors in common (other than 1), so $s - r$ must be divisible by a. But that is impossible, as $0 < s - r < a$.

Therefore all possible remainders appear, and every number from $(a - 1)b$ onwards (and so certainly from ab onwards) appears in the table, so is composable.

We can also generalise the observation at the end of the previous part: the largest non-composable number is $(a - 1)b - a = ab - b - a = (a - 1)(b - 1) + 1$.

> You might wish to think about what happens if the condition that a and b have no common factor is removed. Where was this condition assumed in the solution?

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Archimedes

Sheet 6

Solutions and comments

This programme of the Mentoring Scheme is named after Archimedes of Syracuse (287–212 BCE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Archimedes.html for more information.

These questions may be used freely within your school or college. You may, without further permission, post them on a website that is accessible only to staff and students of the school or college, print out and distribute copies within the school or college, and use them in the classroom. If you wish to use them in any other way, please consult us.
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University of Leeds, Leeds LS2 9JT

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Version 1.1, Mar 2020

1. In some base between 2 and 9 inclusive, $x = 11$ and $y = 100101$ are prime.

What is this base, and what are x and y in decimal? You must show you have ruled out all bases that do not work.

You can find a quick introduction to number bases at https://nrich.maths.org/1368.

ANSWER base 2, $x = 3$, $y = 37$

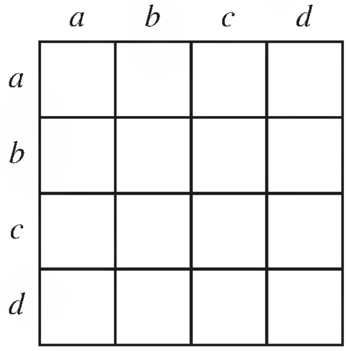
SOLUTION

We lay out our working in a table. We check the value of x first (which is the easier one to calculate); this eliminates all but three potential bases.

Decimal value of base	Decimal value of x	Decimal value of y
2	3	37
3	4	
4	5	1041
5	6	
6	7	7813
7	8	
8	9	
9	10	

1041 is divisible by 3 and, relatively easily, 7813 is divisible by 13. Hence the only possible base is 2, with $x = 3$ and $y = 37$.

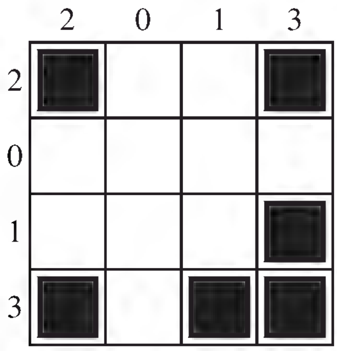
2. The cells of this 4 by 4 grid are to be coloured black or white. The label to the left of each row is the number of black cells in that row; the label above each column is the number of black cells in that column. In the case that $a = 2$, $b = 0$, $c = 1$, $d = 3$ (corresponding to the year 2013), find the solution. If the numbers are taken from any year between 2000 and 2099 inclusive in the same way, how many years have a solution?



ANSWER 13

SOLUTION

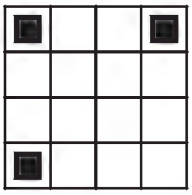
This is the diagram for year 2013:

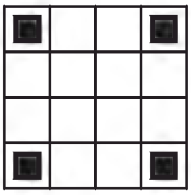


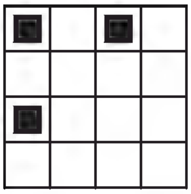
We first note that the year 2000 is impossible since three rows and three columns have to be white. In 200x, the two middle rows and columns have to be white, so only the years 2001 and 2002 can be considered. Both are possible.

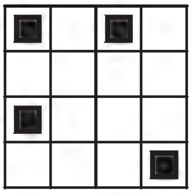
In the rest of the years, the second row and second column have to be white, so no year may contain a digit exceeding 3. Hence any year after 2033 is automatically ruled out. We also note that 2030 is impossible because it contains two zeros.

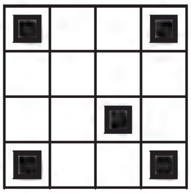
All the others are possible as shown; some of them can also be achieved in other ways.

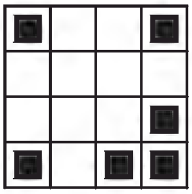

2001

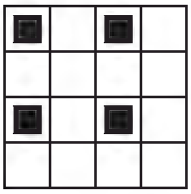

2002

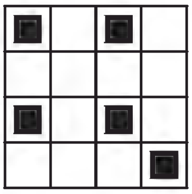

2010

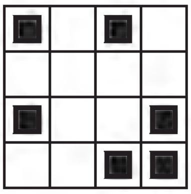

2011

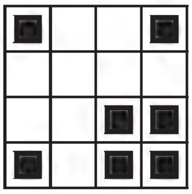

2012

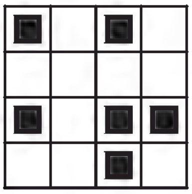

2013

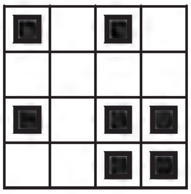

2020

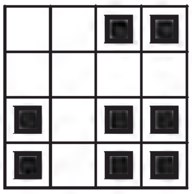

2021


2022

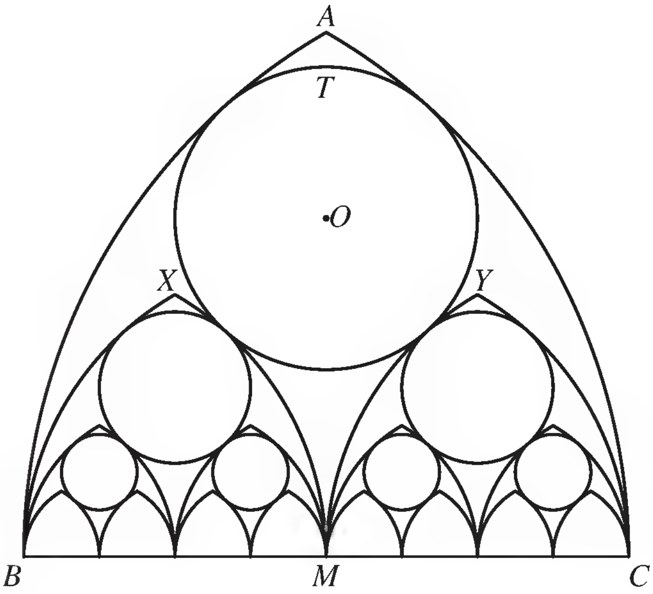

2023


2031


2032


2033

3. The diagram shows more detail of the arch design encountered on the previous sheet. The design is repeated at successively smaller scales. The skeleton shown is assumed to consist of lines with zero width whereas the stone which surrounds the windows is obviously not so.



[Leave π and simplified square roots in your answers.]

(a) Let $OT = 1$. Calculate the perimeter of the shape $ABXMYCA$ formed of six circular arcs.

(b) Calculate the perimeter of the shape $AB\ldots CA$ where the points represented by ... follow the eight smallest arches betwen B and C, giving 2 large and 16 small arcs in total.

(c) Calculate the area enclosed within the perimeter ABC formed by the two large circular arcs.

(d) Notice that each set of arches and circles is a scaled down version of a larger set of arches and circles. Show that the area within $ABXMYCA$ is half the overall area of ABC. Let the area below the smallest set of arches shown be R. If another smaller set of arches is added, what will the area below the smaller arches be, in terms of R?

Answer (a) $\frac{16}{3}\pi$; (b) $\frac{16}{3}\pi$; (c) $\frac{16}{3}\pi - 4\sqrt{3}$; (d) $\frac{R}{2}$

Solution

By *length of an arch* as used below, we shall mean the total length of the two arcs forming the arch.

(a) Each arc is $\frac{1}{6}$ of a circle.

The arc AB has radius 4 (as we saw on the previous sheet) and so has length $\frac{1}{6} \times 2\pi \times 4 = \frac{4}{3}\pi$. So the upper arch has length $\frac{8}{3}\pi$.

The arcs BX, XM, MY, YB each have radius 2, giving the total length of these two arches as $4 \times \frac{1}{6} \times 2\pi \times 2 = \frac{8}{3}\pi$. We add these to find that the perimeter of $ABXMYCA$ is $\frac{16}{3}\pi$.

(b) Each of the arcs in the 8 small arches has radius $\frac{1}{2}$, so the total of the smaller arch lengths is $16 \times \frac{1}{6} \times 2\pi \times \frac{1}{2} = \frac{8}{3}\pi$, which is the same as the length of the two larger arches in the previous part. The total perimeter is thus unchanged by using the smaller arches.

(c) Shape ABC can be divided into two congruent shapes, ABM and ACM (with AB and AC curved). The area ABM can be obtained by subtracting the area of $\triangle AMC$ from the area of sector ABC with centre C and radii CA and CB. Let h be the height of $\triangle AMC$. By Pythagoras's Theorem, $h^2 + 2^2 = 4^2$ so $h = \sqrt{12} = 2\sqrt{3}$.

Alternatively, you may recall that the ratio of sides in half an equilateral triangle is $1 : \sqrt{3} : 2$; this is shown by Pythagoras's Theorem as we have just done.

We can now calculate areas:

$$\begin{aligned}\text{area of } \triangle AMC &= \tfrac{1}{2} \times 2 \times 2\sqrt{3} = 2\sqrt{3}\\ \text{area of sector } ABC &= \tfrac{1}{6} \times \pi \times 4^2 = \tfrac{16}{6}\pi\\ \text{so area of } ABM &= \tfrac{16}{6}\pi - 2\sqrt{3}\end{aligned}$$

We double this to give the area of ABC, namely $\frac{16}{3}\pi - 4\sqrt{3}$.

(d) The arch XBM has half the side length of the arch ABC because $BM = 2$ and $BC = 4$. Since it has half the side length, the area within arch XBM is $\frac{1}{4}$ of the area within arch ABC. (When calculating the area, the calculation is identical in structure to that in part (c), except that all of the lengths are halved. Since the area is calculated by multiplying two lengths together, the area is one-quarter of that under ABC.)

As arch YCM is congruent to arch XBM, the two arches together contain half the area of arch ABC. Thus the area within $ABXMYCA$ is half the area within arch ABC. The area below the smallest arches is halved as another row of smaller arches is added, by the same reasoning, so the area is $\frac{R}{4}$.

> If the pattern were repeated indefinitely, the result would be what is called a fractal. The areas $ABXMA$, $ACYMA$, BXM and CYM are all equal (where the MA line is straight). As extra arches are added, the perimeter remains $\frac{16}{3}\pi$ because it remains jagged along the bottom. However, the perimeter of the shape ABC with BC straight is $\frac{8}{3}\pi + 4$, so the shape with lesser area has larger perimeter.

4. What is the largest integer that is a divisor (factor) of $(n+2)(n+3)(2n+5)$ for every integer n?

ANSWER 6

SOLUTION

Note that it is not enough to show that some integer d is always a divisor; we must also show that there is no larger such integer.

Trying small integers, we find that when $n = -1$, the expression equals 6, so the largest integer that is a divisor of $(n+2)(n+3)(2n+5)$ is a divisor of 6. Some further experiments suggest that this largest integer may be exactly 6.

Another way of seeing that the largest possible common divisor is 6 is to substitute $n = 0$, giving $2 \times 3 \times 5$, and $n = 1$, giving $3 \times 4 \times 7$; these have a greatest common divisor (highest common factor) of 6.

Now $n+2$ and $n+3$ are consecutive integers, so one of them is even, and hence 2 is always a factor. Now consider whether the expression is a multiple of 3. We can write $2n + 5 = 2(n + 1) + 3$. If either of $n + 2$ or $n + 3$ is a multiple of 3, then the whole expression is. If neither is, then we must have $n + 2 = 3k + 1$ and $n + 3 = 3k + 2$ for some integer k, and then $2n + 5 = (n + 2) + (n + 3) = (3k + 1) + (3k + 2) = 6k + 3$, which is a multiple of 3. Thus the expression is always a multiple of 3.

We have therefore proved that the expression is always divisible by 2 and by 3 whatever the value of n, and hence by 6.

Therefore the largest integer that is a divisor of $(n + 2)(n + 3)(2n + 5)$ for every integer n is 6.

The question can also be solved using modular arithmetic, by trying all possible values of n modulo 3, say, to show that the expression is always a multiple of 3, and likewise modulo 2. Combining this with our observation that $n = -1$ gives a value of 6 is enough to solve the problem.

> An alternative approach is to rewrite the expression as $m(m + 1)(2m + 1)$ where $m = n + 2$, which is somewhat simpler to deal with. There can be no higher common prime divisors with just these 3 linear algebraic factors. For example, working modulo p with $p > 3$, taking $m = 1$ gives $1 \times 2 \times 3$, and neither 2 nor 3 is a multiple of p.

5. (a) Expand and simplify $(x + y)(x - y)$.

(b) Prove that every integer can be written in the form $a^2 + b^2 - c^2$ for suitable integers a, b and c.

SOLUTION

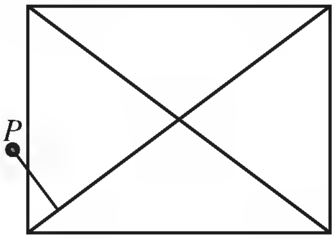
(a) $(x + y)(x - y) = x^2 - xy + yx - y^2 = x^2 - y^2$.

(b) Let $n = a^2 + b^2 - c^2$. Using the previous result, we can write $n = a^2 + (b + c)(b - c)$. We notice that the expression $b - c$ is 1 if $b - c = 1$, which makes the expression for n much simpler. So let us try $b = c + 1$; this gives $n = a^2 + 2c + 1$.

The expression $2c + 1$ can be made to equal any odd integer if c is chosen suitably. Hence if n is odd, we can take $a = 0$, $c = \frac{n-1}{2}$ and $b = \frac{n+1}{2}$, while if n is even, choosing $a = 1$ gives $2c + 1 = n - 1$, so we can take $c = \frac{n}{2} - 1$ and $b = \frac{n}{2}$.

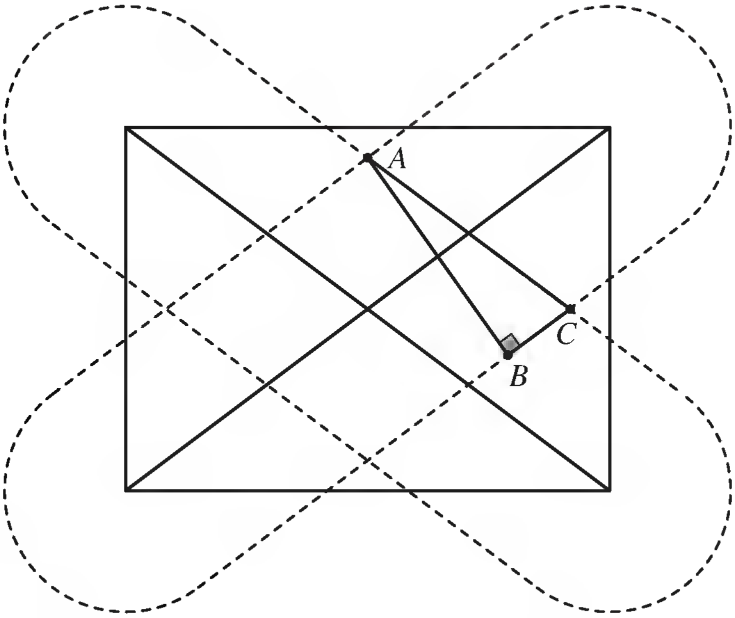
By choosing other even or odd values for a, we can see that there are an infinite number of ways of choosing a, b and c for any given n.

6. A rectangle has diagonals of length 4 units which intersect at an acute angle of 60°. A region is defined by taking all the points at most 1 unit from the nearest diagonal. See the diagram for one such point placed 1 unit from a diagonal. Find the area of the region.



ANSWER $16 - \frac{8}{3}\sqrt{3} + 2\pi$

SOLUTION



The shape is like two athletics tracks which overlap at an angle of 60°, the overlap being a rhombus. (An athletics-track shape is sometimes called a *discorectangle*.)

Each 'track' is a 4-by-2 rectangle capped by two semicircles of radius 1.

Thus each track therefore encloses an area of $8 + \pi$, but the total area is less than twice this, as the overlapping rhombus would be counted twice.

To work out the area of this rhombus, formed by lines 2 apart and containing angles of 60° and 120°, we can drop a perpendicular from the vertex A of the rhombus to the opposite side, meeting it at B as shown on the diagram. Note that $\angle ACB = 60°$ from the information about the diagonals of the rectangle. Thus $\triangle ABC$ is half an equilateral triangle (though it does not appear to be so in our inaccurate sketch).

Now AB is the height of the rhombus, which we know to be 2; we can use this to work out the length AC, which is a side of the rhombus. We can either quote results about the side lengths and height of an equilateral triangle, or derive it as follows. Let $x = AC$. Then $BC = \frac{1}{2}x$, and by

Pythagoras's Theorem we have:

$$\begin{aligned} & x^2 = 2^2 + (\tfrac{1}{2}x)^2 \\ \text{so} \quad & x^2 = 4 + \tfrac{1}{4}x^2 \\ \text{giving} \quad & \tfrac{3}{4}x^2 = 4 \\ \text{hence} \quad & x^2 = \frac{16}{3} \end{aligned}$$

Therefore the length of side of the rhombus is $\frac{4}{\sqrt{3}}$, which may also be written as $\frac{4}{3}\sqrt{3}$ (on multiplying the numerator and denominator by 3). Multiplying the base of the rhombus by its height, we obtain the area of the rhombus, which is $\frac{8}{3}\sqrt{3}$.

The required area is therefore $2(8 + \pi) - \frac{8}{3}\sqrt{3}$.

7. (a) Show that the set of divisors (factors) of 120 can be split into three sets with equal sum.

(b) Show that the sum of the divisors can be written $(8 + 4 + 2 + 1)(3 + 1)(5 + 1)$. How does this relate to the prime factorisation of 120?

(c) Suppose that p is prime and m is a positive integer. Show that the sum of the divisors of p^m can be written as

$$\frac{p^{m+1} - 1}{p - 1}$$

(d) Now suppose that p and q are primes, and that m and n are positive integers. Explain why the sum of the divisors of $p^m q^n$ can be written as

$$\frac{p^{m+1} - 1}{p - 1} \, \frac{q^{n+1} - 1}{q - 1}$$

Solution

(a) The divisors of 120 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 30, 40, 60, 120, and they sum to 360. Therefore to split them into three sets with equal sum, each set must have sum 120. They can be split as follows, for example:

$$1 + 2 + 3 + 4 + 5 + 6 + 8 + 10 + 12 + 15 + 30 = 20 + 40 + 60 = 120$$

There are other ways to do this.

(b) Obviously you could add the divisors and show that it comes to $15 \times 4 \times 6$. However notice two features. The first is that the numbers in each bracket are powers of each prime (including the zeroth power which is 1) which divide into 120. The second is that picking one number from the first bracket, one from the second and one from the third and multiplying the three numbers always gives one of the divisors. Doing this in all arrangements yields *all* the divisors.

(c) The sum of divisors of p^m is $p^m + p^{m-1} + \cdots + p + 1$. Take $m = 3$ to see what happens when we multiply by $p - 1$:

$$\begin{aligned} p^3 + p^2 + p + 1)(p - 1) &= p^4 + p^3 + p^2 + p - (p^3 + p^2 + p + 1) \\ &= p^4 + p^3 + p^2 + p - p^3 - p^2 - p - 1 \\ &= p^4 - 1 \end{aligned}$$

$$\text{Hence} \qquad p^3 + p^2 + p + 1 = \frac{p^4 - 1}{p - 1}$$

We can now see how this will work with any value of m, showing that this sum gives the expression stated.

(d) Again take an example with $m = 3$ and $n = 2$. Every divisor of p^3q^2 has the form (power of p) times (power of q), where the power of p is at most 3 and the power of q is at most 2.

The clue in the expression given is that we should multiply $p^3 + p^2 + p + 1$ by $q^2 + q + 1$. We obtain:

$$(p^3q^2 + p^2q^2 + pq^2 + q^2) + (p^3q + p^2q + pq + q) + (p^3 + p^2 + p + 1)$$

from which we can see that each of the separate terms, for example p^2q^2, will divide into p^3q^2, and that we have listed all the divisors in this way. Hence the sum of all divisors is

$$(p^3 + p^2 + p + 1)(q^2 + q + 1).$$

It is clear that the same argument will work for any values of m and n. The resulting expression is the same as the given expression by applying part (b) to each bracket.

> A necessary condition for it to be possible to split the divisors into 3 sets with equal sum is that the sum of the divisors is divisible by 3. To see how a systematic search might be made for further cases, we could draw up a table of sums of divisors of powers of prime numbers p^m, such as one below where multiples of three are shown in italics.

	$p = 2$	$p = 3$	$p = 5$	$p = 7$	$p = 11$	$p = 13$	...
$m = 1$	*3*	4	*6*	8	*12*	14	...
$m = 2$	7	13	31	*57*	133	*183*	...
$m = 3$	*15*	40	*156*	400	*1464*	2380	...

> We still have to check that the partition can be done. For example, from this table we can see that the sum of the divisors of $24 = 2^3 \times 3$ is $15 \times 4 = 60$, which is divisible by 3. However, 24 itself is more than a third of 60, so it is not possible to split the factors of 24 into three equal-sum sets. The reader might like to consider whether the divisors of any number with just two prime factors can be partitioned in this way.

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Archimedes

Sheet 7

Solutions and comments

This programme of the Mentoring Scheme is named after Archimedes of Syracuse (287–212 BCE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Archimedes.html for more information.

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Version 1.0, April 2019

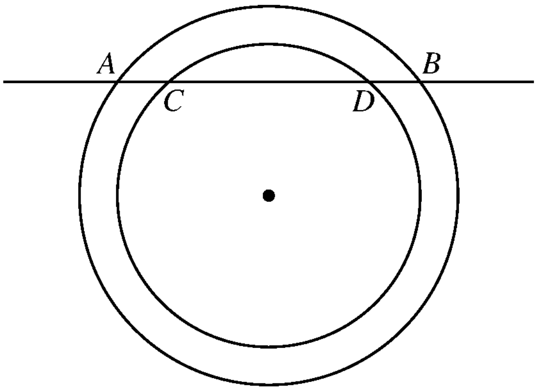
1. What are the integer solutions to the equation $(x+1)^{2-x} = 1$?

Answer $-2, 0, 2$

Solution

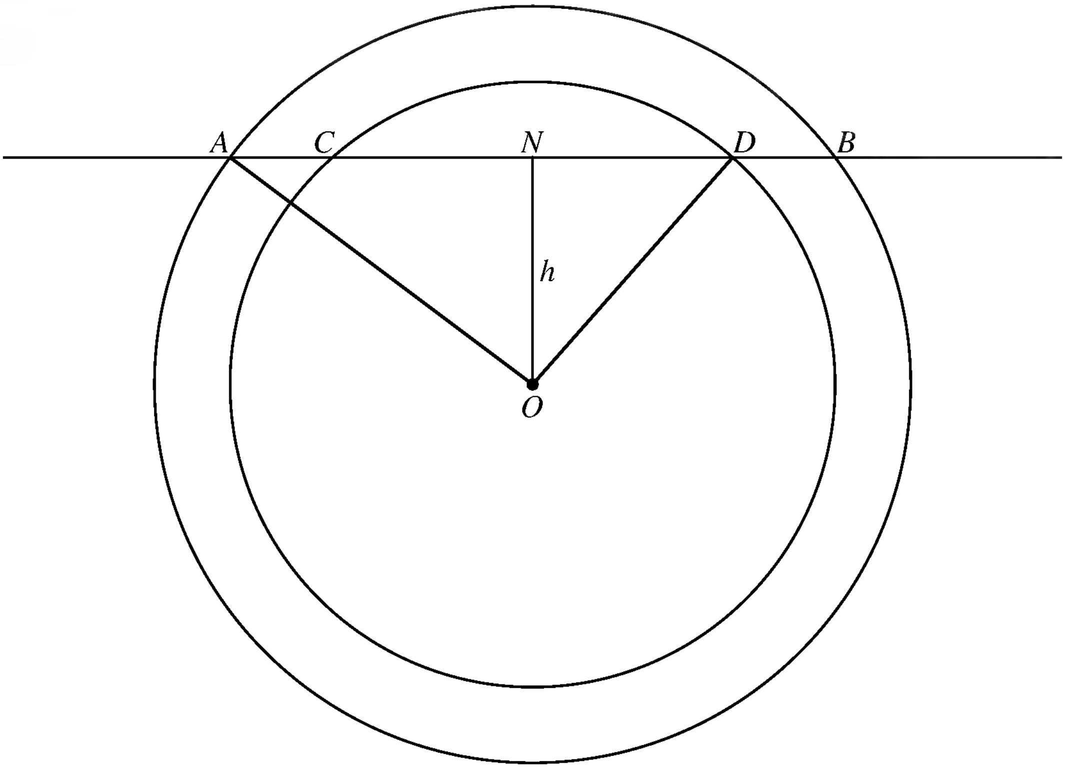
There are various cases:
(a) $x+1$ is -1 with an even power. This requires $x = -2$ which does make the power $2 - x$ even.
(b) $x+1$ is 1. This is the case when $x = 0$.
(c) $2 - x$ is 0. This is the case when $x = 2$.

2. Two concentric circles (i.e. with the same centre) have radii 4 and 5. A straight line cuts the outer circle at A and B and the inner circle at C and D. Given that $AB = 2 \times CD$, how long is AB?



Answer $4\sqrt{3}$

Solution



Let h be the distance of the common centre O from the chord at N (as marked) and $4L$ the length of the longer chord AB. The length of the shorter chord will be $2L$. Using $\triangle ANO$ and $\triangle DNO$, we can write down two Pythagoras equations:

$$\begin{aligned} h^2 + (2L)^2 &= 5^2 \\ h^2 + L^2 &= 4^2 \\ \text{Subtracting these two equations gives :} \quad 3L^2 &= 9 \\ L &= \sqrt{3} \end{aligned}$$

AB is 4 times this length.

3. Without using a calculator, simplify $(1013 \times 2021 \times 1010 + 1006) \div 1012^2$.

Answer 2019

Solution

Taking $x = 2012$, we can write the numerator as:

$$\begin{aligned} (x+1)(2x-3)(x-2) + x - 6 &= (x+1)(2x^2 - 3x - 4x + 6) + x - 6 \\ &= (x+1)(2x^2 - 7x + 6) + x - 6 \\ &= (2x^3 - 7x^2 + 6x + 2x^2 - 7x + 6) + x - 6 \\ &= (2x^3 - 5x^2 - x + 6) + x - 6 \\ &= 2x^3 - 5x^2 \end{aligned}$$

The denominator is x^2 so the division gives $2x - 5$ which gives $2 \times 1012 - 5$.

4. Consider the set of integers from 1 to 2019. How many sets of numbers can be chosen from this set such that the sum of the chosen numbers is 2039185?

[You are not allowed to use a number more than once per set.]

Answer 3

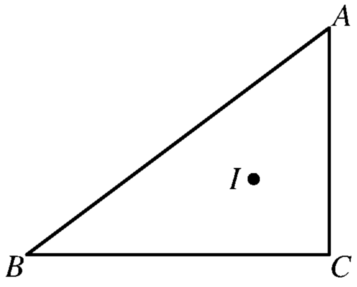
Solution

The problem is much simplified by counting which numbers should be left out. The integers from 1 to 2019 add up to $\frac{1}{2} \times 2019 \times 2020 = 2039190$, the next triangle number exceeding 2039185. We only have to leave out a total of 5.

This can be done in three ways, namely 5 by itself, $1 + 4$ or $2 + 3$.

> 2039190 is the nearest triangle number. If you want to see more about how triangle numbers are formed, try the following links to websites: http://nrich.maths.org/2274 and http://www.mathsisfun.com/algebra/triangular-numbers.html

5. $\triangle ABC$ is right-angled at C. $BC = 20$ and $AC = 15$. I is the centre of a circle (inside the triangle) which touches all three sides of $\triangle ABC$ and has radius r. Find an expression for the area of $\triangle BIC$ in terms of r. Using similar expressions for the areas of $\triangle CIA$ and $\triangle AIB$, find r.
Show that if p is the perimeter of *any* triangle $\triangle ABC$ (right-angled or not) and r is the radius of its inscribed circle, then the area of $\triangle ABC$ is $\frac{1}{2}pr$.

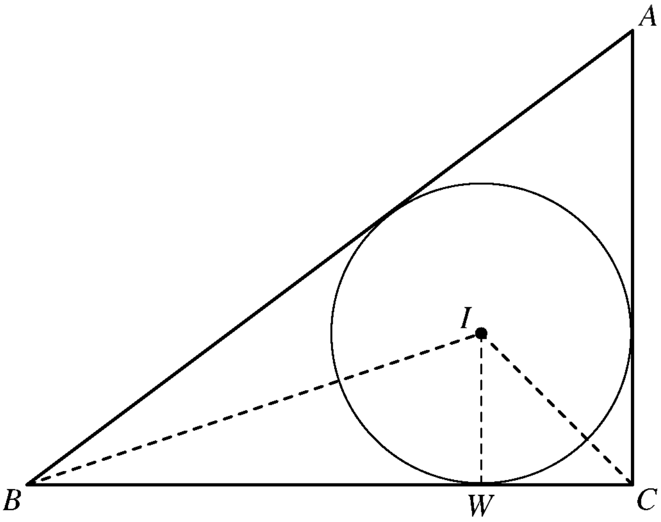


N.B. It is advisable when doing any sort of geometry question to sketch the figure reasonably accurately.

ANSWER $r = 5$

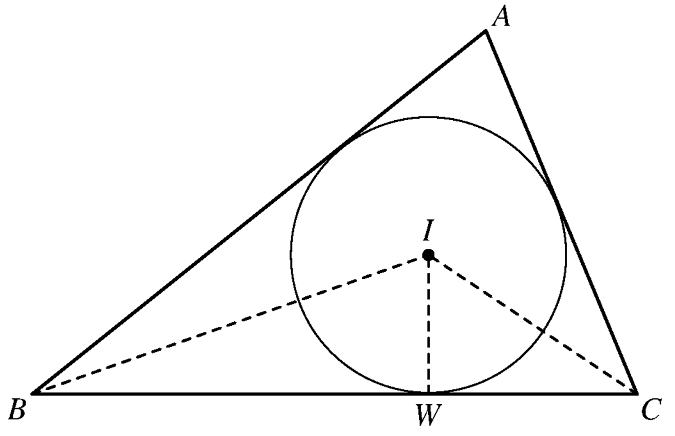
SOLUTION

From Pythagoras's Theorem, $AB^2 = AC^2 + BC^2 = 625$ so $AB = 25$.



The height of each triangle is r because if the circle touches BC at W, then IW is at right angles to BC.
The area of $\triangle BIC = \frac{1}{2} \times 20 \times r = 10r$.
Similarly the area of $\triangle CIA = \frac{1}{2} \times 15 \times r = 7.5r$ and the area of $\triangle AIB = \frac{1}{2} \times 25 \times r = 12.5r$.
Thus the area of $\triangle ABC = 30r$ by putting these three triangles together.
But the area of $\triangle ABC = \frac{1}{2} \times base \times height = \frac{1}{2} \times 20 \times 15 = 150$. Hence $r = 5$.

The diagram below shows a non right-angled triangle with the centre of the touching circle joined to the vertices. Let $BC = a, CA = b$ and $AB = c$.
The area of $\triangle BIC = \frac{1}{2} \times a \times r = \frac{1}{2}ar$. The results for $\triangle BIC$ and $\triangle CIA$ are similar.
Hence the area of $\triangle ABC = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$.
Since $p = a + b + c$, the result is proved.



> Notice the conventional labelling that the side given length a is opposite the vertex A, and so on.
>
> The circle which touches the sides of a triangle on the inside of the triangle is called the *incircle*. Its centre I is the common meeting point of the bisectors of the angles at the vertices. Why must all three lines meet in one point?

6. This question is about perfect numbers. 6 is perfect. Its divisors less than itself are 1, 2 and 3. They add up to itself.

(a) Verify that the next such number is 28.

(b) Using the method of prime factorisation or otherwise to find divisors, show 496 is perfect.

(c) If $2^n - 1$ is prime, show that $2^{n-1}(2^n - 1)$ is perfect.

SOLUTION

(a) For 28, the divisors 1, 2, 4, 7 and 14 (less than 28) add to 28. Some (tedious) calculations will show that no number between 7 and 27 inclusive is perfect.

(b) $496 = 2^4 \times 31^1$.
It derives 5 divisors 1, 2, 4, 8, 16 from its prime factor 2. Each of these can be multiplied by 31, giving 31, 62, 124, 248 and 496 itself, namely another 5 divisors.
We note that, starting at 1, the sum of successive powers of 2 is 1 less than the next power of 2.
Excluding 496, the divisors add to :
$(1 + 2 + 4 + 8 + 16) + 31 \times (1 + 2 + 4 + 8) = 31 + 31 \times 15 = 31 \times 16 = 496.$

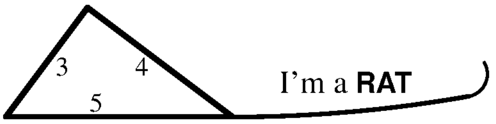
(c) In the more general case of $2^{n-1}(2^n - 1)$, the divisors add to:

$$\begin{aligned}
&(1 + 2 + \cdots + 2^{n-1}) + (2^n - 1)(1 + 2 + \cdots + 2^{n-2}) \\
&= (2^n - 1) + (2^n - 1)(2^{n-1} - 1) \\
&= (2^n - 1) \times 1 + (2^n - 1) \times (2^{n-1} - 1) \\
&= (2^n - 1) \times 2^{n-1} \quad \text{which is the original number.}
\end{aligned}$$

$2^n - 1$ must be prime otherwise the number would have other divisors besides these.

> For $2^n - 1$ to be prime, n must be prime. However if n is prime, it does not necessarily follow that $2^n - 1$ is prime. For example, $2^{11} - 1 = 23 \times 89$. Numbers of the form $2^n - 1$ which *are* prime are called Mersenne primes.
>
> Perfect numbers constructed this way are even. The question of whether there are *any* odd perfect numbers remains unsolved.

7. For the purposes of this topic, we will define a **RAT** to be a right-angled triangle with integer side lengths.



Note that if the sides have fractional (rational) values for their lengths, we can always obtain integer values by using a smaller unit. Thus the set [0.15, 0.2, 0.25] using a particular unit becomes the set [3, 4, 5] if we use a unit of length twenty times smaller. You cannot do this in cases such as a right-angled triangle with sides $[1, 1, \sqrt{2}]$.

(a) Given a **RAT** with lengths a, b and c such that $a^2 + b^2 = c^2$ and such that there is no common factor between all three of a, b and c, prove that just one of a and b is even and that c is odd.

(b) If m and n are positive integers ($m > n$), prove that the set of values $[m^2 - n^2, 2mn, m^2 + n^2]$ can be used to construct a **RAT**.

(c) We will call m and n the key numbers for the triangle (abbreviated to **KN**). What are the **KN**s for these five **RAT**s with lengths of sides as given by each set of three integers?

(i) [3, 4, 5]

(ii) [5, 12, 13]

(iii) [8, 15, 17]

(iv) [20, 21, 29]

(v) [15, 20, 25] . . . careful!

Which two **RAT**s are the same shape? Do they both have a pair of **KN**s?

ANSWER (c) (m, n) = (i) (2, 1), (ii) (3, 2), (iii) (4, 1), (iv) (5, 2), (v) it is impossible to find **KN**s

SOLUTION

(a) If a and b are both even, then c is even and there will be a common factor of 2.

Now consider a and b both to be odd and let $a = 2k + 1$.
Then $a^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1$ so a^2 is one more than a multiple of 4. The sum of two odd squares must be 2 more than a multiple of 4 so c^2 is even. But if $c = 2r$, then $c^2 = 4r^2$ and is a multiple of 4, that is, a contradiction. Hence a and b can not *both* be even.

Hence c odd and one of a or b is odd.

(b) We take $a = m^2 - n^2, b = 2mn, c = m^2 + n^2$ and expand the brackets:

$$\begin{aligned} a^2 &= (m^2 - n^2)^2 \\ &= (m^2 - n^2)(m^2 - n^2) \\ &= m^4 - m^2n^2 - m^2n^2 + n^4, \\ &= m^4 - 2m^2n^2 + n^4, \\ b^2 &= (2mn)^2 \\ &= 4m^2n^2, \\ c^2 &= (m^2 + n^2)^2 \\ &= (m^2 + n^2)(m^2 + n^2) \\ &= m^4 + 2m^2n^2 + n^4. \end{aligned}$$

We now see that the three numbers fit $a^2 + b^2 = c^2$ which shows they satisfy the requirements of Pythagoras's Theorem.

(c) The triangle represented by (v) is 5 times larger than the one represented by (i) : (i) and (v) are proportional sets. The values of m and n do not turn out to be integers in (v) because 5 is not a square. Note that we have not proved that the formula in (b) will give every **RAT**, only that there are **RAT**s which can be obtained in this way. This is important if we choose to confine **KN**s to integers.

8. You are given a 7 by 7 grid of unit squares, 29 of which are black in no particular pattern, the rest white. Call a 2 by 2 square which can be placed exactly over any 4 of these squares a *block*. Prove that a block can be found covering at least 3 black squares.

Solution

For convenience we use the word *cell* to represent a single square on the board in order to avoid ambiguous use of the word square. There are 36 positions in which you could place a block on the board. Let us cover some preliminary thoughts.

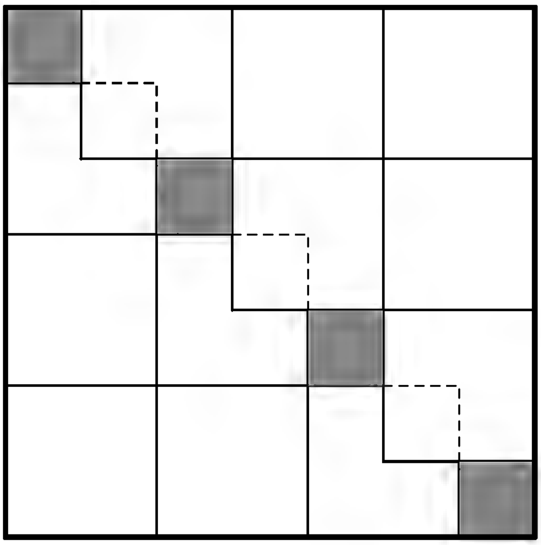
If the cells are coloured like a chess board with 25 black cells and 24 white cells (or vice versa), then we can place a block anywhere on the board so that it covers exactly 2 black cells. We can not find a position where the block covers 3 black cells.

What if there are more black cells? Consider the board covered with lines of black cells. We could have 28 black cells and we could still place a block anywhere so that it covers no more than 2 black cells. If we add a 29th black cell on the board, there must be at least one block that covers three black cells.

However we still have not considered all patterns of black cells, namely that *every* arrangement with 29 black cells will force into existence a block that covers more than 2 black cells. Since there are 29 black cells, there are 49 − 29 = 20 white cells.

Assume we do have an arrangement where each block covers at least 2 white cells and at most 2 black cells. Consider the 12 positions of blocks on the board as shown in the diagram. These blocks must cover at least 24 white cells in total. Because 3 cells are shared by 3 pairs of blocks, and these 3 cells might be white, we can deduce that there must be *at least* 21 white cells covered by these blocks.

This is in contradiction to the fact that only 20 white cells are available.



29 is the critical number of cells in this problem. What would it be if the grid size is $2k + 1$ by $2k + 1$ cells?

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Archimedes

Sheet 8

Solutions and comments

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See http://www-history.mcs.st-and.ac.uk/Biographies/Archimedes.html for more information.

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1. I have 64 cubes, each with edge length 1 cm. 20 of the cubes are white and 44 are blue. If I glue them all together to form a cube with edge length 4 cm, what is the smallest possible surface area that is white?

ANSWER 12 cm^2

SOLUTION

8 cubes can be hidden entirely. This leaves 12 white cubes which must appear as part of the surface. There are $6 \times 4 = 24$ places for cubes which only show one face, so this is where these 12 cubes should be placed. All the rest of the cubes with a face on the surface will be blue.

2. Two circles touch each other externally at T. One has centre A and radius 80, the other has centre B and radius 90. A tangent from B to the other circle touches that circle at D. The common tangent to both circles at T cuts BD at C. How long is CD?

If circles 'touch externally', it means that the circles touch but do not lie inside each other.

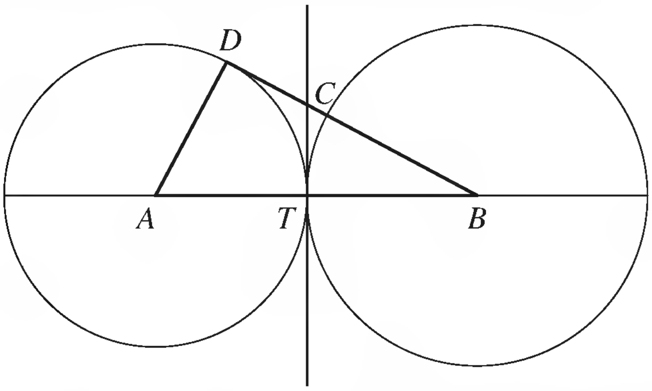
ANSWER 48

SOLUTION

We begin by drawing a clear diagram of the situation, as shown below. We observe that $\angle ADB = 90°$ since AD is a radius to the point of contact of the tangent at D. Using Pythagoras's Theorem, we then have:

$$\begin{aligned} & AD^2 + BD^2 = AB^2 \\ \text{so} \quad & 80^2 + BD^2 = 170^2 \\ \text{giving} \quad & BD^2 = 22500 \end{aligned}$$

so $BD = 150$. (You may recognise (8, 15, 17) as a Pythagorean triple.)



We observe that $\triangle CTB$ is similar to $\triangle ADB$ because $\angle ABD$ is a common angle to both triangles, and $\angle CTB$ and $\angle ADB$ are both right angles. Hence:

$$\frac{BC}{BT} = \frac{BA}{BD} \quad \text{so} \quad \frac{BC}{90} = \frac{170}{150}.$$

This gives $BC = 102$. Hence $CD = 48$.

An alternative approach is as follows. We observe that $CD = CT$ because these are a pair of tangents to the circle centre A. (We can prove this by showing that $\triangle ATC$ is congruent to $\triangle ADC$.)

Having proved that $BD = 150$, we can now let $CD = x$ and apply Pythagoras's Theorem to $\triangle BTC$, giving

$$\begin{aligned} && (150 - x)^2 &= x^2 + 90^2 \\ \text{expanding gives} && 22500 - 300x + x^2 &= x^2 + 8100 \\ \text{hence} && 14400 &= 300x \end{aligned}$$

This again gives $x = 48$.

3. Here is an example of five consecutive positive integers whose sum is 1000:

198, 199, 200, 201, 202

Find the largest possible number of consecutive positive integers whose sum is exactly 1000 and justify your answer.

ANSWER 25

SOLUTION

If the first consecutive integer is F and the last integer is L, the mean of the integers is $\frac{1}{2}(F + L)$. If there are n integers, then their total is $\frac{1}{2}n(F + L)$, which we require to equal 1000, so $n(F + L) = 2000$.

We must also have $n < \frac{1}{2}(F + L)$, for otherwise the sequence includes zero or negative integers. Multiplying this inequality by n gives $n^2 < \frac{1}{2}n(F + L)$, so $n^2 < 1000$, hence $n < 32$.

We note that $L - F = n - 1$ as there are n numbers in the sequence. So if we know $F + L$ and $n - 1$, we can add them to obtain $2L$, and hence find both L and F.

So we can now consider factor pairs of $2000 = 2^4 \times 5^3$, where both factors are positive, and deduce the corresponding values of L and F. We are looking for the largest possible value of n, so we start with $n = 25$, being the largest factor of 2000 less than 32. This gives $F + L = 80$ and $L - F = 24$, hence $2L = 104$. This gives $L = 52$ and $F = 28$, and so this value of n is valid.

A similar approach is as follows. Note that $n = L - F + 1$, so if n is odd, then $F + L$ is even, so $\frac{1}{2}(F + L)$ must be an integer. On the other hand, if n is even, then $F + L$ is odd, so $\frac{1}{2}(F + L) - \frac{1}{2}$ is an integer. We also note that n must be less than $\frac{1}{2}(F + L)$, as before, and we require $\frac{1}{2}n(F + L) = 1000$.

Consider, then, the factors pairs $[n, \frac{1}{2}(F + L)]$ of 1000, including numbers of the form $k + \frac{1}{2}$ for $\frac{1}{2}(F + L)$:

$\underline{1 \times 1000}$, 2×500, 4×250, 8×125, $\underline{5 \times 200}$, 10×100, $\underline{16 \times 62\frac{1}{2}}$, 20×50, $\underline{25 \times 40}$ and 40×25.

Further factor pairs will certainly have n too large. Only the underlined factor pairs satisfy these requirements with $n = 1$, 5, 16 and 25. (Of course, it is doubtful whether $n = 1$ satisfies the requirement for consecutive integers!)

4. If x is the solution to the equation:

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}=x,$$

show that it satisfies $x^2-x-1=0$.

Show that if x is a solution to a similar equation with one more level of fractions, then x still satisfies $x^2-x-1=0$.

SOLUTION

On the left-hand side, work from the inside of the fraction outwards, repeatedly combining fractions by writing them over a common denominator or by taking their reciprocals:

$$\begin{aligned}
1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{x}}} &= 1+\cfrac{1}{1+\cfrac{1}{\left(\frac{x+1}{x}\right)}} && \text{combining bottom-most fractions}\\
&= 1+\cfrac{1}{1+\cfrac{x}{x+1}} && \text{taking the reciprocal}\\
&= 1+\cfrac{1}{\left(\frac{2x+1}{x+1}\right)} && \text{combining bottom-most fractions}\\
&= 1+\frac{x+1}{2x+1} && \text{taking the reciprocal}\\
&= \frac{3x+2}{2x+1}
\end{aligned}$$

Another way of simplifying the left-hand side is to multiply the numerator and denominator of the compound fraction. For example, we can simplify $\cfrac{1}{1+\frac{1}{x}}$ by multiplying the numerator and denominator by x, giving $\dfrac{x}{x+1}$. Using this approach, the simplification runs as follows:

$$1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{x}}} = 1+\cfrac{1}{1+\frac{x}{x+1}} \qquad \text{multiplying bottom fraction by } \frac{x}{x}$$

$$= 1+\frac{x+1}{x+1+x} \qquad \text{multiplying main fraction by } \frac{x+1}{x+1}$$

$$= 1+\frac{x+1}{2x+1}$$

$$= \frac{(2x+1)+(x+1)}{2x+1} \qquad \text{combining into one fraction}$$

$$= \frac{3x+2}{2x+1}$$

Equating this fraction to x gives us the equation $3x+2 = x(2x+1)$. This rearranges and simplifies to

$$x^2 - x - 1 = 0$$

as required.

If the initial fraction is continued to one more level, that is,

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}} = x,$$

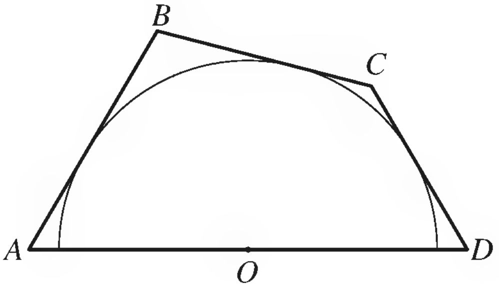
a simplification like the above leads to

$$\frac{5x+3}{3x+2} = x$$

which again yields the equation $x^2 - x - 1 = 0$.

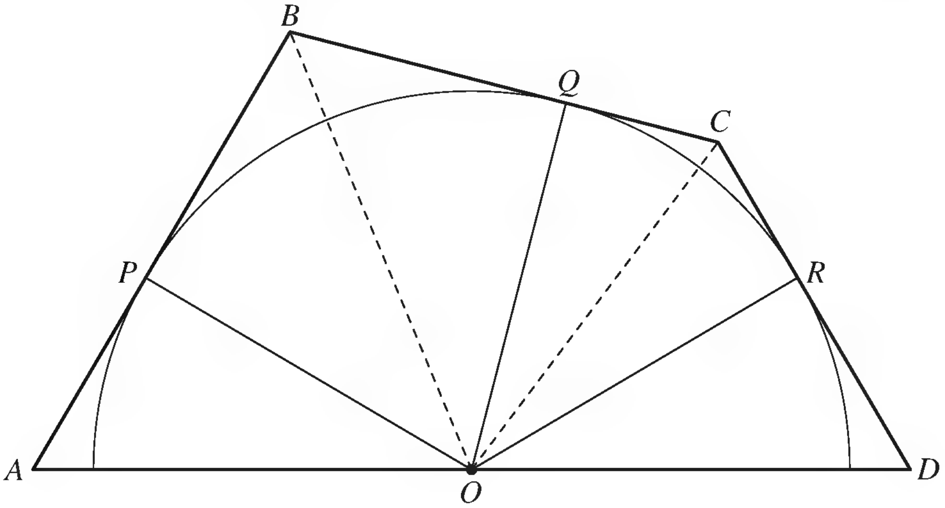
> You might have noticed the Fibonacci numbers appearing as coefficients in these simplified fractions; this is not a coincidence, and extending the original *continued fraction* to further levels gives simplifications $\frac{8x+5}{5x+3}$, $\frac{13x+8}{8x+5}$ and so on. They all lead to the same quadratic equation.
>
> By constructing a suitable table of values of $x^2 - x - 1 = 0$ for integer values of x, it can be seen that there is just one positive value of x, lying between 1 and 2, that satisfies the equation. This solution is the golden ratio.

5. A quadrilateral $ABCD$ circumscribes a semicircle. The diameter and centre O lie on AD. You are given that $AO = OD$. Prove that $AO^2 = AB \times CD$.



Solution

In the diagram P, Q and R are the points of contact of the sides with the semicircle.



Since $AO = OD$ (given), using $OP = OR$ (radii) and $\angle OPA = \angle ORD = 90°$, we can prove $\triangle ODR$ is congruent to $\triangle OAP$. Hence $\angle OAB = \angle ODC$.

Since the lengths of the two tangents to a circle from an exterior point are equal, we have $BP = BQ$ and $CQ = CR$. There are thus two further pairs of right-angled congruent triangles: $\triangle OBP$ is congruent to $\triangle OBQ$ and $\triangle OCQ$ is congruent to $\triangle OCR$.

The sum of the angles around O is 180°. Thus $\angle AOP + \angle POB + \angle COR = 90°$ (taking one of each pair of equal angles). But $\angle OCR + \angle COR = 90°$, so $\angle AOP + \angle POB = \angle OCR$.

We therefore have $\angle AOB = \angle DCO$, and as $\angle BAO = \angle ODC$, it follows that $\triangle AOB$ is similar to $\triangle DCO$.

[Note the ordering of the letters, which indicates how the vertices match up. You can think of it as rotating $\triangle DCO$ anticlockwise around O and then sliding it to the left.]

It follows that $\frac{OA}{CD} = \frac{AB}{OD}$, which on multiplying by $OD \times CD$, and noting that $OD = OA$, gives $OA^2 = AB \times CD$ as required.

> Students might like to note that they can use the symbol $\cong$ as shorthand for the relationship 'is congruent to' and $\sim$ for 'is similar to'.

6. In this question, n is a positive integer. Which of the following statements are true?

(a) *If* $3n + 1$ is an odd square, *then* n is a multiple of 8.

(b) It is *necessary* for n to be a multiple of 4 for $3n + 1$ to be an odd square.

(c) It is *sufficient* that n is a multiple of 4 for $3n + 1$ to be an odd square.

(d) $3n + 1$ is an odd square *if and only if* n is a multiple of 8.

ANSWER (a) True; (b) True; (c) False; (d) False

SOLUTION

It may be worth noticing that $3n + 1$ is a square for $n = 5$ but this square is not odd. All the statements relate to $3n + 1$ being an *odd* square. An odd number can be written as $2k + 1$ for some integer k, and an odd square can be written as $(2k + 1)^2$ for some integer k.

(a) Suppose that $3n + 1$ is an odd square. Then $3n + 1 = (2k + 1)^2$ for some integer k.

Expanding gives $3n + 1 = 4k^2 + 4k + 1$, so $3n = 4k(k + 1)$. Since one of k and $k + 1$ must be even, it follows that $3n$ is a multiple of 8. Because the highest common factor of 3 and 8 is 1, n itself must therefore be a multiple of 8.

(b) Since n must be a multiple of 8, it is necessarily a multiply of 4.

(c) It is easy to see that this is not the case. Consider $n = 4$. This gives $3n + 1 = 13$, which is not an odd square.

(d) Part (a) shows that *if* $3n + 1$ is an odd square, *then* n is a multiple of 8. We need to determine whether the *converse* is also true: *if* n is a multiple of 8, *then* $3n + 1$ is an odd square. Trying $n = 8, 16, 24, 32, \ldots$ quickly shows that this is not the case: when $n = 24$, $3n + 1 = 73$, which is not an odd square. In conclusion, while the *only if* part of the statement is true, the *if* part is false, and so the whole statement is false.

7. A plane is tesselated with equilateral triangles. Colour one of these triangles and call it T. A polygon P_1 is formed by colouring all the triangles which have contact with sides or vertices of T. Show that there are 13 coloured triangles within the perimeter of P_1.

Polygon P_2 is formed by colouring all triangles which have side or vertex contact with P_1, polygon P_3 is formed by colouring all triangles which have side or vertex contact with P_2, and so on. How many triangles are coloured within the perimeter of P_{10}?

ANSWER 661

SOLUTION

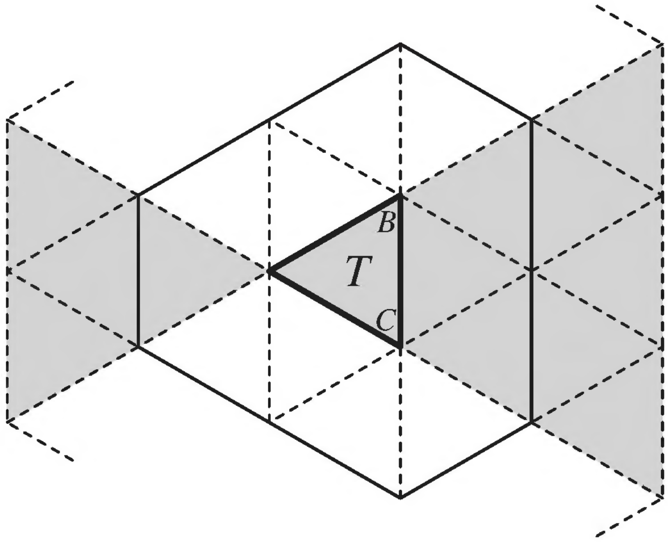
Observe that as triangles are added, the shape of P_n remains a hexagon. You will find it helpful to draw the case up to at least P_2 (partly shown in the figure below) to understand what is going on.

Consider two edges of T passing through the vertex A extended in both directions. These contain two triangular regions of P_n as shown in the figure for P_2. In the direction including T, the number of triangles is the $(n+1)$th square number; in the other direction it is the nth square number. (Why is this the case? In the smaller direction there are $1+3+5+\cdots+(2n-1)$ triangles; we could now use the reasoning from section B of the notes for mentees on the UKMT Mentoring Scheme webpage to sum this series.)

Hence for P_{10} we have 121 triangles in the direction including T and 100 in the opposite direction. Subtracting 1 to exclude T gives 220 triangles between the lines through A.

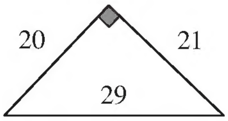
There are two more similar cases with lines through B and through C. A check shows that this includes every triangle within P_n exactly once, except for T itself.

Including one more triangle to account for T, we therefore have $3 \times 220 + 1 = 661$ triangles in total.



> In general the number of triangles between two lines in total excluding T is $(n+1)^2 + n^2 - 1$.
>
> This simplifies to : $2n^2 + 2n = 2n(n+1)$. The final formula is then $6n(n+1)+1$.

8. This question follows on from sheet 7 question 7, about **RAT**s. This is an example of a Nearly Isosceles Right Angled Triangle (**NIRAT**), which is a **RAT** (a right-angled triangle with integer length sides) whose two short sides differ in length by exactly one:



Let $x = m^2 - n^2$, $y = 2mn$ and $z = m^2 + n^2$, where m and n are integer **KN**s.

(a) Show that $x - y = (m - n)^2 - 2n^2$ (Equation 1) and $x - y = 2m^2 - (m + n)^2$ (Equation 2).

(b) If $x - y = 1$ or $y - x = 1$, then $[x, y, z]$ represents a **NIRAT**. Find the **KN**s of these **NIRAT**s: $[3, 4, 5]$ and $[20, 21, 29]$.

(c) These are the two smallest **NIRAT**s. Find the next smallest **NIRAT**.

(d) Show that using the **KN**s from the triple $[21, 20, 29]$ in equation 2, you can find two square numbers such that twice one of them is 1 more or 1 less than the other. Use this result in equation 1 to find two new **KN**s for a larger **NIRAT**, and so find this **NIRAT**. (You should get the same answer as in part (c).)

(e) Prove that if (m, n) are the **KN**s for a **NIRAT** and (M, N) are the **KN**s for the next largest **NIRAT** found using this method, then $M = 2m + n$, $N = m$.

Solution

(a) Starting from the right hand sides, we find

$$\begin{aligned}(m - n)^2 - 2n^2 &= m^2 - 2mn + n^2 - 2n^2\\ &= m^2 - n^2 - 2mn\\ &= x - y2m^2 - (m + n)^2 \qquad\qquad = 2m^2 - (m^2 + 2mn + n^2)\\ &= 2m^2 - m^2 - 2mn - n^2\\ &= m^2 - n^2 - 2mn\\ &= x - y.\end{aligned}$$

(b) Recalling that x is odd and y is even if the side lengths do not have a common factor, we find that the **KN**s for $[3, 4, 5]$ are $m = 2$, $n = 1$ and for $[20, 21, 29]$ are $m = 5$, $n = 2$. Note that these both have $(m - n)^2 - 2n^2 = \pm 1$.

(c) A straightforward but time-consuming approach is to calculate $x^2 + (x + 1)^2$ for $x = 21, 22, 23, \ldots,$ until some value of x gives a perfect square. A more sophisticated approach is as follows.

The side lengths of a **NIRAT** cannot have any factor in common greater than 1, otherwise the shorter sides would differ by more than one. So we can work with the **KN**s of this **NIRAT**, and using equation 1, we find that they must satisfy $(m - n)^2 - 2n^2 = \pm 1$, as x and y must differ by exactly 1. So we are looking for two square numbers $(m - n)^2$ and n^2 with the larger being 1 more or 1 less than twice the smaller. It is not hard to see that 49 and 25 satisfy this condition and that there are no other smaller pairs which have not already been used.

Hence $(m - n)^2 = 49$ and $n^2 = 25$, giving $m = 12$ and $n = 5$. The resulting **NIRAT** is $[119, 120, 169]$.

(d) The **KN**s for $[21, 20, 29]$ are $m = 5$, $n = 2$, giving $2m^2 - (m + n)^2 = 50 - 49 = 1$. Thus our two square numbers are 5^2 and 7^2. In equation 1, this would require $M - N = 7$ and $N = 5$ (using capital letters to distinguish these from our current values of m and n), so $M = 12$ and $N = 5$, making $X - Y = 49 - 50 = -1$.

Use these in the formulas for the side lengths, we obtain the **NIRAT** $[12^2 - 5^2, 2 \times 12 \times 5, 12^2 + 5^2] = [119, 120, 169]$.

(e) What we have done in previous part is to use the old m and $m + n$ from equation 2 and make these the new N and $M - N$ in equation 1. So $N = m$ and $M - N = m + n$, giving $M = 2m + n$ as required.

We now have an *algorithm* for generating a new pair of **KN**s to produce a larger **NIRAT**.

> The **NIRAT**s get rapidly larger. The next **KN**s are $m = 29$, $n = 12$ yielding sides $[696, 697, 985]$. We note that this method produces all **NIRAT**s, but we have not proved that none are left out. To do that, we could take the argument of this question and work backwards: given a **NIRAT**, we can produce a smaller **NIRAT** by reversing the above process. By doing this repeatedly, we can show that we must end up with the **NIRAT** $[3, 4, 5]$, and so the **NIRAT** we started with must have been one of those constructed by this method.

9. Write down the **KN**s you obtained from question 8 in a table including an extra column for s where $s = m + n$.

(a) Check that $s^2 = 2m^2 \pm 1$ for these **NIRAT**s. (This means that for each **NIRAT**, either $s^2 = 2m^2 + 1$ or $s^2 = 2m^2 - 1$.)

(b) If the numbers in a chosen row are denoted by m and s and in the next row by M and S, find M and S in terms of m and s.

(c) Show that $2M^2 - S^2 = -(2m^2 - s^2)$. Hence prove that $s^2 = 2m^2 \pm 1$ for every **NIRAT** constructed in this way.

(d) Show that the isosceles triangle with sides $[m, m, s]$ is approximately a right-angled triangle.

(e) Investigate the values of the fractions $\frac{s}{m}$ in relation to the value of $\sqrt{2}$. Can you explain how the value $\sqrt{2}$ emerges here?

The values (m, s) are solutions to *Pell's equation*, an equation dating back to late Greek mathematics. The fractions $\frac{s}{m}$ are called *continuants* of $\sqrt{2}$.

Answer (b) $M = s + m$, $S = s + 2m$

Solution

(a) Here are the first few **NIRAT**s, including two further ones beyond those found in the previous question.

m	n	s	$2m^2$	s^2
2	1	3	8	9
5	2	7	50	49
12	5	17	288	289
29	12	41	1682	1681
70	29	99	9800	9801

We can easily see that $s^2 = 2m^2 \pm 1$ for each of the rows of this table.

(b) We use the results from part (e) of the previous question:

$$M = 2m + n = m + (m + n) = m + s$$
$$S = M + N = (2m + n) + m = 2m + (n + m) = 2m + s$$

(c) Using the relations from the previous to substitute for M and S, we find:

$$\begin{aligned} 2M^2 - S^2 &= 2(m+s)^2 - (2m+s)^2 \\ &= 2m^2 + 4ms + 2s^2 - 4m^2 - 4ms - s^2 \\ &= s^2 - 2m^2 \\ &= -(2m^2 - s^2). \end{aligned}$$

Since $s^2 - 2m^2 = +1$ in the first row, it has the value -1 in the second, $+1$ in the third and so on, so $s^2 = 2m^2 \pm 1$ for every **NIRAT** constructed in this way.

(d) We have $s^2 = 2m^2 \pm 1 \approx 2m^2$, so $s^2 \approx m^2 + m^2$ and the sides of the triangle almost obey the conditions for Pythagoras's Theorem.

We can be a lot more precise than this if we wish, using more advanced mathematics. We can ask what the vertex angle, θ, of this triangle is, and we can answer this using the cosine rule. We have

$$s^2 = m^2 + m^2 - 2m^2 \cos\theta$$

so that

$$\cos\theta = \frac{2m^2 - s^2}{2m^2} = \frac{2m^2 - (2m^2 \pm 1)}{2m^2} = \mp\frac{1}{2m^2}.$$

So as m increases, $\cos\theta$ tends to zero, and thus the vertex angle tends to a right angle.

(e) The fractions become closer and closer to $\sqrt{2}$.

Dividing the equation $s^2 = 2m^2 \pm 1$ by m^2 gives $(\frac{s}{m})^2 = 2 \pm \frac{1}{m^2} \approx 2$ because $\frac{1}{m^2}$ is very small. Therefore $\frac{s}{m} \approx \sqrt{2}$.

Furthermore, as m increases in size, the approximation improves, and $\frac{s}{m}$ approaches $\sqrt{2}$.

> An interested student might like to consider the continued fractions that were investigated in question 4. What sequence of continued fractions converges to the value $\sqrt{2}$, and how does it relate to the **NIRAT**s in this question?